CALIFORNIA INSTITUTE OF TECHNOLOGY
PHYSICS MATHEMATICS AND ASTRONOMY DIVISION

Freshman Laboratory

Physics 3 Course

Experiments Notes

Academic Year 2004-2005

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Acknowledgments

I started this work with the aim of improving the course of Physics Laboratory for Caltech freshmen students, the so called ph3 course . Thanks to Donald Skelton, ph3 was already a very good course, well designed to satisfy the needs of news students eager to learn the basics of laboratory techniques and data analysis.

Because of the need of introducing new experiments, and new topics in the data analysis notes, I decided to rewrite the didactical material trying to keep intact the spirit of the course, i.e emphasis on techniques and not on the details of the theory.

Anyway, I believe and hope that this attempt to reorganize old experiments and introduce new ones constitutes an improvement of the course.

I would like to thank, in particular, Eugene W. Cowan for his incommensurable help he gave to me with critiques, suggestions, discussions, and corrections to the notes. His experience as professor at Caltech for several years were really valuable to make the content of these notes suitable for students at the first year of the undergraduate course.

I would like to thank also all the teaching assistants that make this course work, for their patience and valuable comments that I constantly received during the academic terms. Christophe Basset deserves a particular mention for his effort which went beyond his assignments.

Sincerely,

Virgínio de Oliveira Sannibale

Contents

Chapter 1

Data Analysis

Sections 1, 3, and 4 of the notes "Vademecum for Data Analysis Beginners" contain the information necessary to complete test of this chapter. It is indeed mandatory to read those sections before starting answering the questions.

Some of the questions require the use of a computer program named "CurveFit", which will be explained during the first laboratory class. Sheets to make graphs will be provided during the class.

Data Analysis Questions

1. The thickness d of the base of a cylindrical can is computed by measuring the outside height $H = (97.3 \pm 0.2)$mm and the inside height $h = (97 \pm 1)$mm. What is d and its uncertainty σ_d? Does the more precise measurement of H affect the uncertainty σ_d?

2. A computer program gives values for two parameters in the following form:

   a = 12.37825 m/s, error on a =0.0286145 m/s
   b = 3.2395e-3 m, error on b = 2.7481912e-05 m

   How do you report these values?

3. A distance x is measured using a ruler of length $l = 300$mm, which is too short for a direct measurement. Using the ruler stepwise nine

times, it is found that the distance x is equal to 2654mm. If the uncertainty of each measurement is $\sigma = 1$mm, what is the uncertainty σ_x on x ?

4. We have two measurements of the same quantity obtained using two different techniques

$$p_1 = (1.231 \pm 0.019)\text{kg/m}^2 \text{ and } p_2 = (1.262 \pm 0.012)\text{kg/m}^2 \ .$$

   Do these two measurements agree?

5. Six measurements are made of the voltage difference across a resistor. The results are as follows:

Measurement n.	1	2	3	4	5	6
Voltage Difference (V)	1.44	1.48	1.47	1.43	1.50	1.47

   Compute the uncertainty of each measurement and the uncertainty of mean value by hand, report the measurements in the proper format, and check your results with a computer program (eg. CurveFit).
   If the resistance is measured to be $R = (15.1 \pm 0.1)\ \Omega$, what is the power $P = \frac{V^2}{R}$ dissipated by the resistor and its uncertainty σ_P?

6. Let $I = \frac{1}{2}M(R^2 + r^2)$. What is the equation for σ_I, if M, R and r are measured quantities? If we require $\sigma_I/I = 0.1\%$, what is the relative precision for the measurements of M , R and r ? (Suppose that $R = \alpha r,\ \alpha > 1$, and $\Delta R = \Delta r$).

7. The voltage V along a transmission line is $V(x) = V_0 \exp(-x/x_a)$, where x is the position along the line. Find x_a and its uncertainty σ_{x_a} Measuring V_0 , V and x. Find the best value of V_0, which minimizes σ_{x_a}.

8. The angle variation θ of a wheel rotating under a constant acceleration, is measured at different times t. A plot of t^2 vs. θ of the data is fitted with a straight line $y = a + bx$. Supposing that the initial angular velocity was zero, what are a and b? What is the value of σ_{t^2} for $t = (1.32 \pm 0.02)$ s? .

9. Plot the following set of data by hand on linear graph paper.

x(s)	0.5	1.0	2.5	3.5	4.5	5.0	6.0
σ_x(s)	0.1	0.1	0.1	0.1	0.1	0.1	0.1
y(cm)	21.0	28.0	41.5	55.0	61.0	70.0	77.5
σ_y(cm)	1.0	1.0	1.0	1.5	2.0	3.0	1.0

Supposing that the best fitting curve of experimental data is a straight line $y = ax + b$, graphically estimate the parameters a, and b.
Fit the data using an appropriate program (eg. CurveFit), and analyze the differences plot. Try to reconcile any significant differences between the program fit parameters and your own.

10. Plot the following data by hand on both linear and semi-log graph paper:

x(a.u.)	16	44	76	87	110
y(a.u.)	0.037	0.097	0.25	0.41	0.80

(Uncertainties on the y values are 6% of the value; i.e. $\sigma_y = 0.06y$.)

Determine a relationship between x and y, and graphically estimate the uncertainties in all the fit parameters.
Plot and fit the data using an appropriate computer program (eg. CurveFit),and compare with your previous results.

Chapter 2

The Maxwell Top

2.1 Introduction

In this chapter we want to study some particular cases of rigid body dynamics, which have a rotational symmetry around one axis, the so-called top, or Maxwell top.[1]

In general, to solve the dynamics of a rigid body we must apply the second law of dynamics, i.e.

$$\frac{d\vec{L}}{dt} = \vec{\tau},$$

where $\vec{\tau}$ is the external torque acting on the body, and $\vec{L}$ is its angular momentum. For a solid body rotating around one of its axis of symmetry $\hat{z}$ (more generally, around any of its three principal axes), with angular velocity $\dot{\theta}$, $\vec{L}$ is given by[2]

$$\vec{L} = I\dot{\theta}\hat{z}, \tag{2.1}$$

where I is the moment of inertia around the $\hat{z}$ axis. I is

$$I = \int (x^2 + y^2) dm,$$

where x and y are the coordinates of the mass dm. In this particular case,

[1]The study of the top general equations of motion is quite complicated and is one of the main topics of a classical mechanics course.

[2]The dot above the symbol stands for the derivative with respect to the time t. The number of dots indicates the order of derivation.

the second law of dynamics assumes the simpler form

$$I\ddot{\theta}\hat{z} = \vec{\tau}.$$

2.2 Some Relevant Examples

In this section we will study three particular cases of the Maxwell top dynamics, which will be used in the laboratory procedures.

2.2.1 Angular Acceleration under a Constant Torque

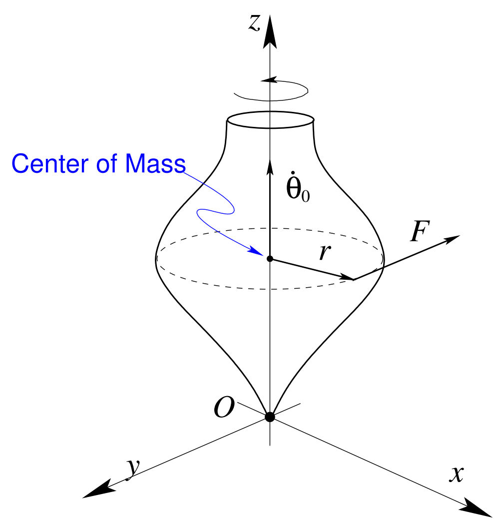


Figure 2.1: Top subject to an external force $\vec{F}$ which produces a torque $\vec{\tau} = rF\hat{z}$.

Let's consider a top, whose axis of symmetry is vertical, and a force $\vec{F}$ applied tangent to the top's surface in the horizontal plane containing the top's center of mass (see figure 2.1). If $\vec{F}$ remains constant in modulus and direction in the reference frame rotating with the top, the second law of dynamics assumes a very simple form, i.e.[3]

$$I\ddot{\theta} = rF,$$

where r is the arm lever distance. Integrating the previous equation we get

$$\theta(t) = \theta_0 + \dot{\theta}_0 t + \frac{1}{2}\ddot{\theta}_0 t^2, \qquad \ddot{\theta}_0 = \frac{rF}{I}. \tag{2.2}$$

where θ_0 is the initial angle, $\dot{\theta}_0$ the initial angular velocity, and $\ddot{\theta}_0$ is the angular acceleration which is also constant.

2.2.2 Top Suspended with a Torsional Rod

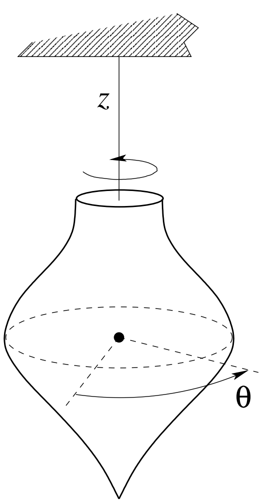


Figure 2.2: Top suspended to a torsional rod.

By suspending the top with a torsional rod (see figure 2.2) we will have a restoring torque (the torsional version of the Hooke's law) given by the linear equation

$$\tau = -k\theta,$$

[3]we are neglecting the energy dissipation mechanisms, which are always present in any physical system.

where θ is the angle in the horizontal plane measured from the equilibrium position. From the second law of dynamics, we will have

$$I\ddot{\theta} = -k\theta,$$

which is the equation of an harmonic oscillator, whose general solution is

$$\theta(t) = \theta_0 \cos(\omega_0 t + \varphi_0), \qquad \omega_0^2 = \frac{k}{I}.$$

The top will oscillate sinusoidally around the vertical axis with angular frequency ω_0. The constant ω_0 is said to be the angular resonant frequency of the torsional pendulum.

2.2.3 Precession of the Top

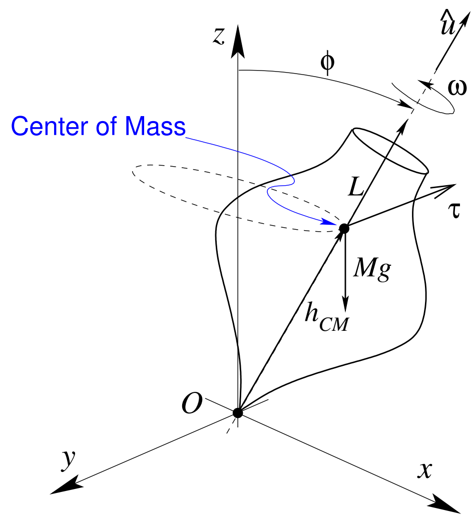


Figure 2.3: Precession of the top. In this sketch $\vec{\tau}$ is parallel to the y axis and is pointing to the negative direction of the y axis. The vector $\vec{h_{CM}}$ is in the plane Oxz

Let's suppose now that the top has its tip constrained on a horizontal plane and is rotating around its axis $\hat{u}$ at a constant angular velocity $\omega\hat{u}$ (see figure 2.3).

If the rotation axis makes an angle ϕ with $\hat{u}$ axis, the modulus of the torque $\vec{\tau}$, due to the gravity force $M\vec{g}$, is

$$\tau = |\vec{h}_{CM} \times M\vec{g}| = h_{CM} M g \sin\phi, \tag{2.3}$$

where $\vec{h}_{CM}$ is the vector pointing to the top center of mass. Because $\vec{g}$ is always vertical, $\vec{\tau}$ must always lie in the horizontal plane.

Because $\omega\hat{u}$ is parallel to $\vec{h}_{CM}$ at all times $\vec{L}$ is parallel to $\vec{h}_{CM}$ also. It follows that $\vec{L}$ is perpendicular to $\vec{\tau}$ at all times.

For the second law of dynamics and because $\vec{\tau}$ is always in the horizontal plane, the variation $d\vec{L}$ of the angular momentum must be always in the horizontal plane. This implies that projection of $\vec{L}$ along the vertical axis is constant and $\vec{L}$ can only rotate about the vertical axis. As a consequence the component of $\vec{L}$ in the horizontal plane is constant in modulus but not in direction.

Considering that the projection of $\vec{L}$ in the horizontal plane is $L\sin\phi = const.$, the variation dL of $\vec{L}$ must be (see figure2.4)

$$dL = L\sin\phi d\alpha,$$

where $d\alpha$ is the infinitesimal angular variation in the horizontal plane.

Using the second law of dynamics and the previous expression, we get

$$L\sin\phi\frac{d\alpha}{dt} = \tau,$$

The derivative is indeed the angular velocity Ω of the top around the vertical axis. Combining the previous expression with the (2.3) we get

$$h_{CM} M g = L\Omega.$$

Substituting the (2.1) into the previous equation ($\dot{\theta} = \omega$) we finally get

$$\Omega = \frac{Mg}{I\omega} h_{CM}, \tag{2.4}$$

which shows that Ω does not depend on the angle ϕ. Ω is said to be the precession angular frequency and when $\Omega \neq 0$, the top is said to precess around the vertical axis. It is worthwhile to notice that the angular momentum modulus $|\vec{L}|$ is conserved.

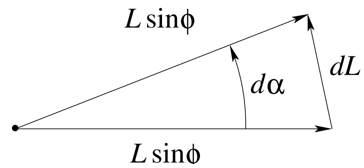


Figure 2.4: Projection and variation of the angular momentum in the horizontal plane

2.3 Experimental setup.

The Maxwell top is shown schematically in Fig.2.5. The top floats on an air cushion which creates a thin "air film" (less than 80μm) and considerably reduces the frictional losses of energy. The two drive jets give the top a small torque, which can be changed acting on the adjustable exhaust valve. The sliding mass m changes the position of the top center of mass along the top axis. Fig.2.5 also shows some details of the air circuit which sustains the top, and creates a thin air film for friction reduction between the base and the top.

Some other instruments needed for the two-week experiment are the following:

- a balance to measure various masses,
- a tachometer to measure the top angular velocity about its axis,
- a ring to increase the top moment of inertia,
- a torsional rod to suspend the top,
- a quasi-frictionless pulley, and a 2g weight to apply a constant torque to the top.

2.3.1 Care and Use of the Experimental Apparatus

The air bearing is a particularly delicate device because of the the air film thickness. Any scratch or dirt on the air bearing surfaces can compromise the use of the experimental apparatus.

These are the precautions that need to be taken:

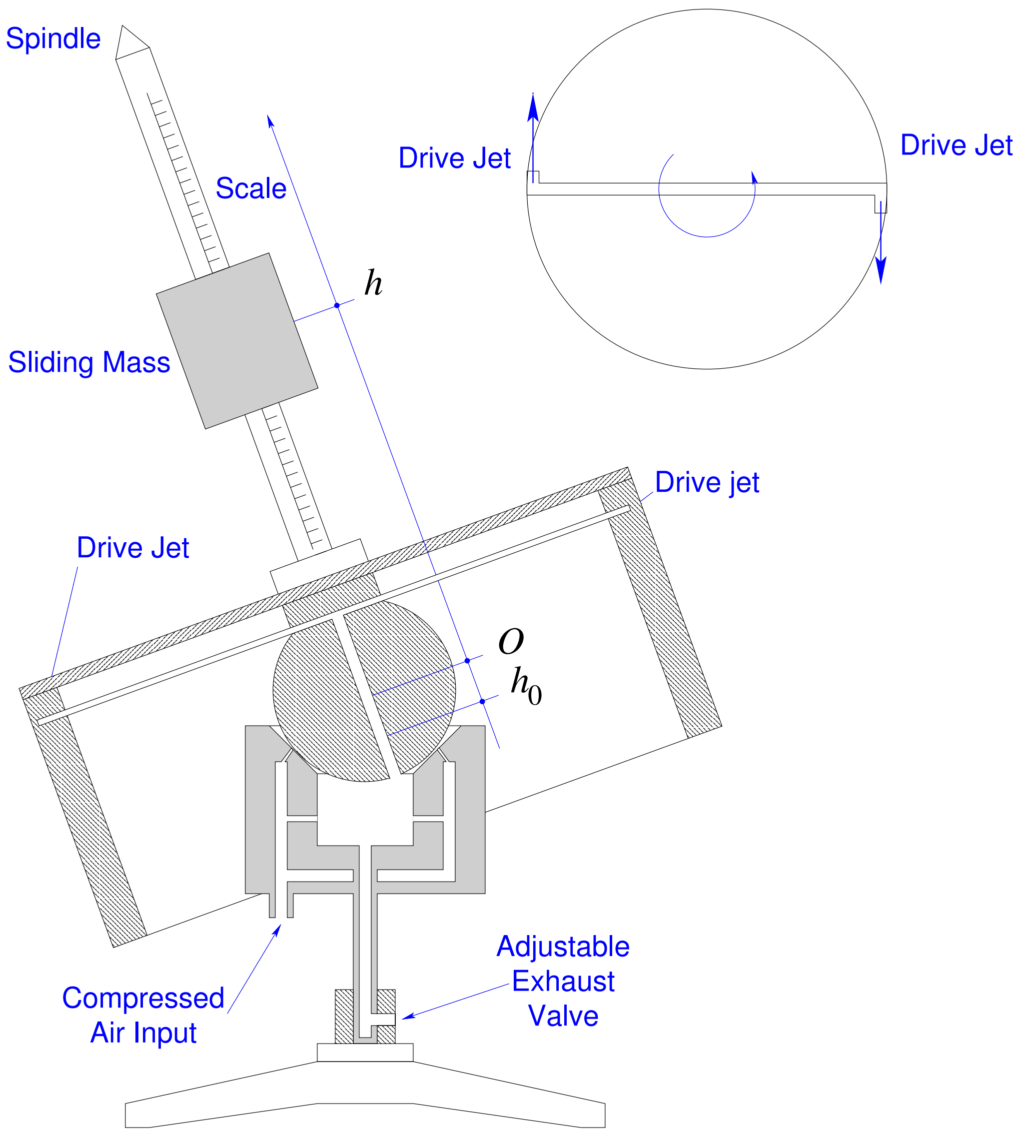


Figure 2.5: Maxwell Top schematic vertical cross section and top view cross section. O is the top's pivoting point and the sketched axis is oriented as indicated by the arrow. This implies that h_0 is negative and h is positive.

- TURN THE AIR SUPPLY TO **26PSI** BEFORE ANY OPERATION.
- NEVER LET THE TOP SIT ON THE AIR BEARING BASE WITHOUT AIR FLOW.
- DO NOT SWITCH TOPS. EACH TOP WORKS PROPERLY WITH JUST ONE BASE.
- DO NOT LET ANY OBJECT FALL DOWN INTO THE AIR BEARING CUP.
- DO NOT USE CLEAR SCOTCH TAPE TO ATTACH WIRES TO TOP'S CYLINDER. USE SCOTCH MASKING TAPE PROVIDED BY THE LABORATORY.
- ALWAYS REMOVE THE SCOTCH TAPE FROM THE TOP'S CYLINDER ONCE FINISHED.
- REMEMBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

2.4 First Laboratory Week

The purpose of the lab is to apply the two methods of measuring the moment of inertia, based on the theory explained in the previous sections, and compare and analyze the results.

2.4.1 Indirect Measurement of the Moment of Inertia Applying a Constant Torque

The top's moment of inertia I can be indirectly measured if we apply a known constant torque rF to it, and measure the revolution time of the top.

In fact, measuring the elapsed time for $1, 2, 3, ..., n$ revolutions, and fitting the data to the equation (2.2), we can obtain the value of the parameter $\ddot{\theta}_0$ and indeed I.

2.4.2 Indirect Measurement of the Moment of Inertia Using a Torsional Pendulum

Another way to make an indirect measurement of a rigid body moment of inertia I is measuring the period T of a torsional pendulum, whose bob is the rigid body itself. By adding to the bob another rigid body with the known moment of inertia I_0 and re-measuring T, it allows to compute I without knowing the characteristic of the torsional rod.

In fact, the angular frequencies of the rotation about the axis of symmetry for the two cases are

$$\omega_1^2 = \frac{k}{I}, \qquad \omega_2^2 = \frac{k}{I + I_0},$$

which combined together, and considering that $\omega_i = 2\pi/T_i$, result in

$$I = \frac{I_0}{\left(\frac{T_2}{T_1}\right)^2 - 1} \tag{2.5}$$

The value of I_0can be obtained indirectly by the definition of moment inertia.

2.4.3 Propaedeutic Problems

1. Derive how the moment of inertia I_0 for a ring of inner radius r, outer radius R, and mass M, is given by

$$I_0 = \frac{1}{2}M(R^2 + r^2) \tag{2.6}$$

2. If the ring has a mass $M = (5.000 \pm 0.003)$kg, an outer diameter $D = (200.0 \pm 0.5)$mm, and an inner diameter $d = (180.0 \pm 0.5)$mm, compute I_0 ,$\sigma_{I_0}$and the relative error σ_{I_0}/I_0.

3. The measurement of the oscillation period of a torsional pendulum with a stopwatch, produces an error due to the experimenter's reaction time. Assuming that this error is $\sigma = 0.05$s, the pendulum period is $T = 2$s, and only one measurement is performed, how many periods must be measured to get a relative error σ_T/T of ± 2%, ± 1%, and ± 0.1%?

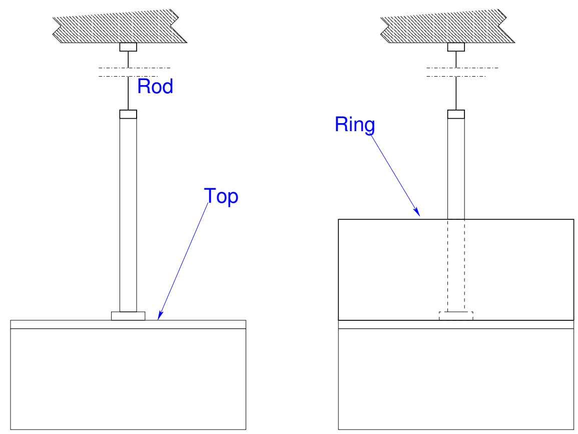
4. In determining the top's moment of inertia I with the torsional pendulum, it is found that the oscillation period is $T_1 = (1.260 \pm 0.003)$s, and with the added ring with moment of inertia I_0, is $T_2 = (1.750 \pm 0.002)$s. Find the uncertainty in the measurement of I. Use the values of I_0 and σ_{I_0}given in problem 2.

5. REMEMBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

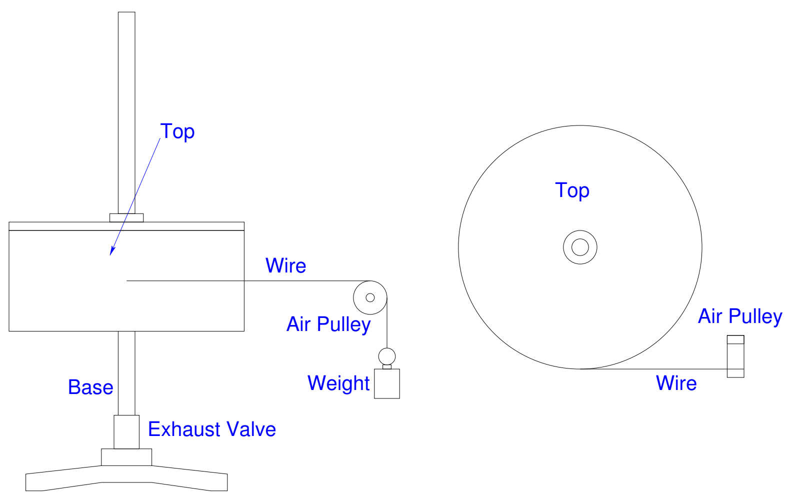
2.4.4 Procedure (Top's Moment of Inertia Measurements)

Remember to follow the directives written in section 2.3.1 (Care and Use of the Experimental Apparatus) before starting the procedure.

1. Determine the top moment of inertia I using the torsional rod as show in the figure below.



2. Determine the top's moment of inertia fitting the angular displacement v.s. elapsed time, when the top is under a constant torque. To realize this condition, attach a string with a 2g weight to the top's cylinder, and run the wire over the air pulley, as shown the figure below.



Adjust the exhaust valve to change the air-jets flow until the top reaches a state closest as possible to the equilibrium. Remove the

string from to the top and measure the revolution periods. To keep the torque as constant as possible, never readjust the air pressure. To obtain a quasi-frictionless pulley, set the air flow of the pulley in such way the free pulley turns at very low speed. Try to keep the top as vertical as possible.
Remember to remove the scotch tape from the rim once finished.

3. Compare the two measured values of the moment of inertia I.

4. Compare the value of the angular acceleration $\ddot{\theta}_0$ obtained from the fit, with the value obtained from the definition of $\ddot{\theta}_0$ using the moment of inertia I calculated in point 1.

2.5 Second Laboratory Week

The purpose of this lab is to verify that the precession angular velocity Ω is independent of the angle ϕ and to study the Ω as a function of the top's center of mass.

If the center of the sliding mass m is placed at a distance h from the top's pivot, and h_0 is the position of the top's center of mass without m, the new center of mass will be located at (see Fig. 2.5)

$$h_{CM} = \frac{h_0 M + hm}{M + m} \tag{2.7}$$

It is important to notice that h_0 is negative because it is below the pivot O, which is the origin of the reference frame chosen to compute h_{CM}. With the addition of the mass m, equation (2.4) becomes

$$\Omega = \frac{(M + m) g h_{CM}}{I\omega}, \tag{2.8}$$

where we have neglected the small increase in the moment of inertia I due to the mass m. Inserting equation (2.7) into the equation (2.8), and after some algebra, we obtain

$$\Omega = \frac{Mg}{I\omega} h_0 + \frac{mg}{I\omega} h. \tag{2.9}$$

which relates the precession angular velocity to the sliding mass position h.

If we impose

$$h = h^* = -h_0 \frac{M}{m}, \tag{2.10}$$

the angular velocity Ω of the precession goes to zero. If the mass m is placed at $h*$, h_{CM} is zero and the torque vanishes, and therefore the top does not precess.

It is important to notice that the spindle length is such that we can change the sign of h_{CM}.

2.5.1 Propaedeutic Problems.

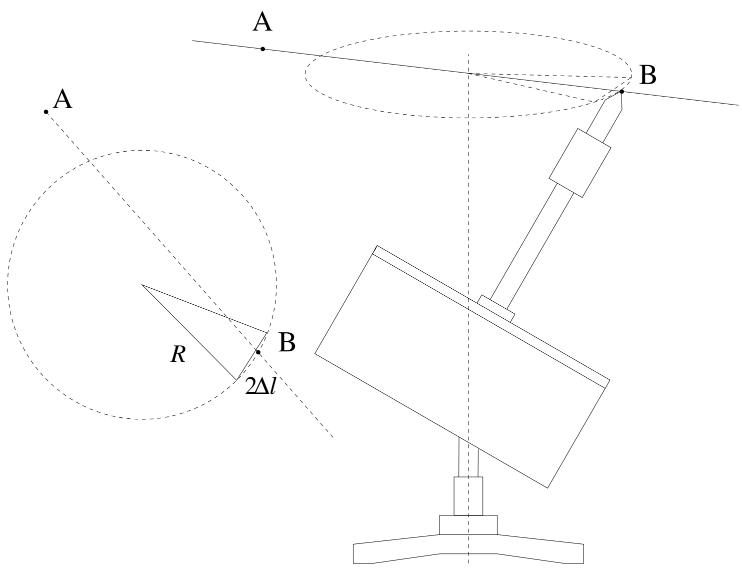
1. The position of the top center of mass h_0 without the sliding mass m, is negative (below the pivot point). Provide a sketch depicting the direction of the angular velocity ω and the direction of the top precession.

2. Calculate the period of precession T for a top spinning at 5Hz (5 revolutions per second) if $m = 0.2186$kg, $I = 4.66 \cdot 10^{-2}$kg m^2, and $h = h^* + 0.01$m.

3. Given the sliding mass $m = 0.2186$kg with its outside diameter $D = 0.033$m, and inside diameter $d = 0.016$m, calculate the sliding mass moment of inertia I_m. Is the statement under equation (2.8) justified?

4. A linear fit to Ω versus h gives $\Omega = a+bh$. What are a and b in terms of m, g, I, ω and h^*? Supposing that ω is constant during each measurement but different every time we change h, how can we rearrange equation 2.9 to still use a straight line as fitting function ?

5. The precession period T is measured with the sliding mass removed, and for a given value of the angular velocity $\omega.$ Write the equation that gives h_0 in terms of ω, T, M, I and g.



6. Using two points, **A** and **B,** to align the line of sight (see figure above), a careful student determines that the uncertainty of measuring where the spindle passes the pointer at **A** is $\Delta l = \pm 2$mm. If the radius of the precession orbit is $R = 50$mm, and the period is $T = 60$s, what uncertainty σ_T does this produce in the period T? What fraction is σ_T of the total period T?

2.5.2 Procedure (Precession Period Measurement)

Remember to follow the directives written in section 2.3.1 (Care and Use of the Experimental Apparatus) before starting the procedure.

Setting the revolution angular velocity ω of the top at around 5Hz make the following measurements:

1. Without the sliding mass, demonstrate that top precession period T is independent from the angle ϕ for constant value of ω.

2. Using the previous measurements of the precession period T , of the angular revolution frequency ω, and of the moment of inertia I calculate $h^*$and its uncertainty . Place the sliding mass at h^* and confirm that the top does not precess.

3. Experimentally study the equation of the precession angular velocity Ω as a function to the sliding mass position h. Be sure that you measure the length of the sliding mass.

4. Compare the new measurement of I obtainable from step 3 with the two ones of the previous week.

5. Calculate the value of h^* obtainable from step 4 and compare it with the previous measurement.

6. REMEMBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

Chapter 3

A Mechanical Oscillator

3.1 Damped Mechanical Oscillator

Consider the mechanical harmonic oscillator sketched in figure 3.1, consisting of a mass attached to two springs, sliding along an air bearing guide (air trough). The mass contains a permanent magnet whose magnetic field closes through the air trough. There will be indeed a viscous damping force generated by the Eddy currents in the metallic guide. For our purposes, it is sufficient to consider the following approximations:

- The springs are one-dimensional and ideal (i.e. they are massless, they obey Hooke's law along the x direction and they are perfectly rigid in other directions, they are not dissipative, etc...).
- The only non-negligible mechanism of energy loss is due to the Eddy current (for example, the air viscosity is negligible compared to the action of the magnetic force).

Naming the spring constants k_1 and k_2, x the coordinate of the mass m, and α the viscous damping coefficient, the equation of motion is

$$m\ddot{x} = -\alpha\dot{x} - k_1 x - k_2 x.$$

Dividing the previous equation by m, rearranging the terms and using the following definitions

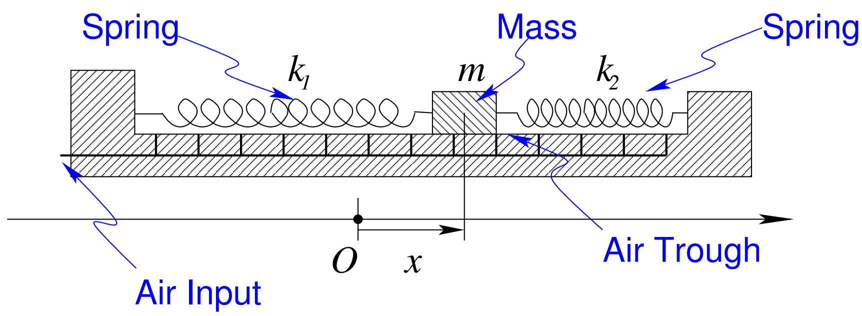


Figure 3.1: Mechanical oscillator sketch

$$\omega_0^2 = \frac{k_1 + k_2}{m}, \qquad \gamma = \frac{\alpha}{m}$$

we obtain

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0, \tag{3.1}$$

which is the well known equation of motion of a damped harmonic oscillator.

DEF.: Resonant Frequency ν_0 of the Undamped Mechanical Oscillator

the quantity $\nu_0 = \omega_0/2\pi$ is said to be the *resonant frequency of the undamped mechanical oscillator.*

3.1.1 Step Response

Substituting the trial function[1]

$$x(t) = \Re\left[Xe^{\lambda t}\right],$$

into equation (3.1), we get the characteristic polynomial equation for λ

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0,$$

[1]The use of a complex function as trial function is just to facilitate the calculations. The physical solution must be real and not a complex function. This explains the presence of the real part symbol $\Re[]$.

with roots

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\Delta}, \qquad \Delta = \gamma^2 - 4\omega_0^2.$$

The general solution of the differential equation of motion is indeed

$$x(t) = \Re\left[X_1 e^{\lambda_1 t} + X_2 e^{\lambda_2 t}\right],$$

where X_1 and X_2 are fixed by the initial conditions at time t_0 (i.e. the position $x(t_0)$ and the velocity $\dot{x}(t_0)$). The behavior of the system depends on the value of the discriminant Δ.

Over-damped Harmonic Oscillator

In the case that $\Delta > 0$, we will have

$$\gamma > 2\omega_0 \quad \Rightarrow \quad x(t) = e^{-\gamma t/2}\left(\Re\left[X_1\right] e^{\sqrt{\Delta}/2t} + \Re\left[X_2\right] e^{-\sqrt{\Delta}/2t}\right).$$

λ_1 and λ_2 are real and the amplitude x decays exponentially.

Critically Damped Harmonic Oscillator

If $\Delta = 0$, we will have

$$\gamma = 2\omega_0 \qquad \Rightarrow \qquad x(t) = e^{-\gamma t/2},$$

λ_1 and λ_2 are real and coincident and the amplitude x decays exponentially.

Damped Harmonic Oscillator

For $\Delta < 0$, we will have

$$\gamma < 2\omega_0 \quad \Rightarrow \quad x(t) = e^{-\gamma t/2}\Re\left[X_1 e^{(i\sqrt{|\Delta|}/2)t} + X_2 e^{-(i\sqrt{|\Delta|}/2)t}\right].$$

with $\lambda_1$and $\lambda_2$complex and conjugate constants.

Defining the following quantity

$$\omega_\gamma^2 = \omega_0^2 - \frac{\gamma^2}{4},$$

extracting the real part, and rearranging the previous equation, we get

$$x(t) = \Re\left[X_1 + X_2\right] e^{-\gamma t/2} \cos(\omega_\gamma t),$$

After some tedious algebra, we finally obtain the exponentially decaying sinusoidal solution

$$x(t) = x_0 e^{-\gamma t/2} \cos(\omega_\gamma t + \varphi_0). \tag{3.2}$$

The amplitude x_0 and the phase φ_0 are defined by the initial conditions (see figure 3.2).

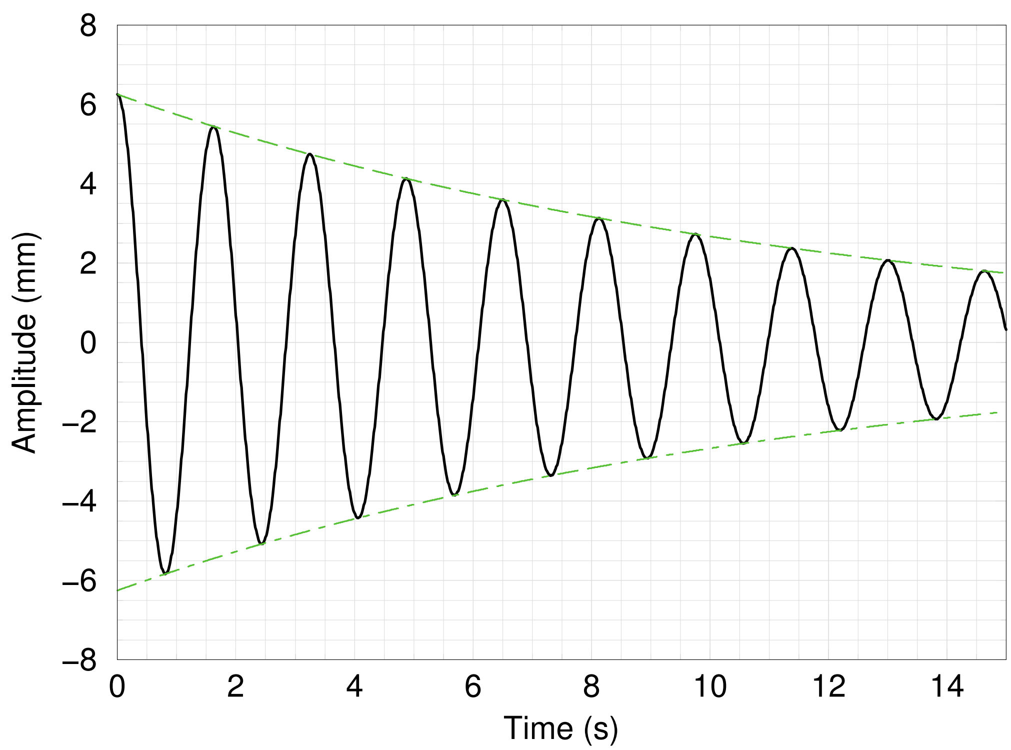


Figure 3.2: Mechanical oscillator ring down. The two exponentially decaying curves are the envelope of the ring down. The parameters used are typical of the real set-up, $\omega_0 = 3.8$ rad/s , $\gamma = .17\ \mathrm{s}^{-1}$, $x_0 = 6.3$ mm.

DEF.:Damped Mechanical Oscillator Resonant Frequency

The constant $\nu_\gamma = \omega_\gamma/2\pi$ is defined as *the resonant frequency of the mechanical damped oscillator*.

DEF.:Mechanical Oscillator Time Constant τ

Defining the following constant as the *time constant* of the mechanical oscillator

$$\tau = \frac{2}{\gamma},$$

the previous equation becomes

$$x(t) = x_0 e^{-t/\tau} \cos(\omega_\gamma t + \varphi_0).$$

We can see from the previous equation that after a time $t = \tau$ the envelope maximum amplitude is reduced by a factor $1/e \simeq 1/2.718$, which means we can easily estimate τ by just measuring the time needed to reduce the initial amplitude x_0 to about $1/3$.

It is worthwhile to notice that the time constant parameter gives a simple way to characterize the behavior of the oscillator. For example, to characterize the amplitude decay we can consider time intervals multiples of τ:

Time (τ)	Initial Amplitude (x_0)	Relative Amp. Respect to the Initial Amp.(%)
1	$\sim 1/3$	37%
3	$\simeq 1/20$	5%
5	$\sim 1/150$	$< 0.6\%$

A crude estimation of τ can be obtained measuring the period T (or the resonant frequency ν_γ), and counting how many periods n^*, the amplitude takes to decrease to $1/3$. The elapsed time will be an estimation of τ, i.e.

$$\tau \simeq T n^* = \frac{n^*}{\nu_\gamma}$$

The uncertainty on the period T, will be about half period $T/2$.

DEF.:Mechanical Oscillator Quality Factor Q

Defining the following constant as the *quality factor* of the mechanical oscillator

$$Q = \frac{\omega_\gamma}{\gamma},$$

equation (3.2) becomes

$$x(t) = x_0 e^{-\omega_\gamma t/(2Q)} \cos(\omega_\gamma t + \varphi_0).$$

Comparing the previous expression with the expression of the time constant we get

$$Q = \frac{\tau\omega_\gamma}{2} = \pi\tau\nu_\gamma,$$

which relates the quality factor to the resonant frequency and to the time constant.

Considering the previous expression, and applying the same method for estimation of τ, we can get a crude estimation of Q which is

$$Q \simeq \pi n^*.$$

3.2 A Forced Mechanical Oscillator

In presence of an external force F, the equation of motion for the mechanical oscillator (re-sketched in figure3.3) becomes

$$m\ddot{x} = -m\gamma\dot{x} - k_1 x - k_2 x + F.$$

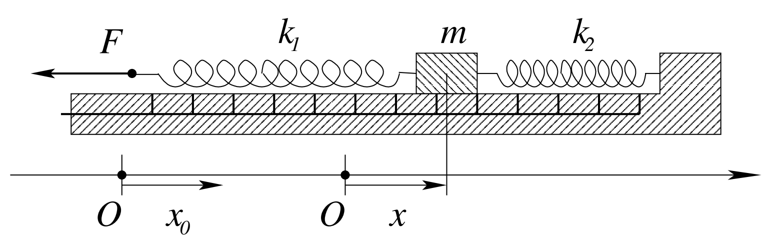


Figure 3.3: Forced Mechanical Oscillator

Rearranging the terms of the previous expression, we obtain the usual form of the equation of motion of the forced oscillator

$$m\ddot{x} + m\gamma\dot{x} + (k_1 + k_2)x = F. \tag{3.3}$$

3.2.1 Solution for Sinusoidal Excitations

If we apply a sinusoidal force F, which produces a displacement $x_0(t) = X_0\Re\left[e^{i\omega t}\right]$ through the spring with spring constant k_1, we will have

$$F = k_1 X_o \Re\left[e^{i\omega t}\right].$$

Substituting this new expression of F into equation (3.3) we get

$$m\ddot{x} + m\gamma\dot{x} + (k_1 + k_2)x = k_1 X_0 \Re\left[e^{i\omega t}\right]. \tag{3.4}$$

In the steady state regime, because of the linearity of the mechanical system[2], we expect the mass to oscillate at the angular frequency ω of the driving force, with amplitude and phase to be determined, i.e.

$$x(t) = \Re\left[X e^{i\omega t}\right],$$

where X is a complex number.

Substituting this trial expression for x into equation (3.4), we get

$$[-m\omega^2 + im\omega\gamma + (k_1 + k_2)]X = k_1 X_0.$$

Dividing by m and defining the following quantities

$$\omega_0^2 = \frac{k_1 + k_2}{m}, \qquad \omega_1^2 = \frac{k_1}{m},$$

we will have

$$(-\omega^2 + i\gamma\omega + \omega_0^2)X = \omega_1^2 X_0,$$

and finally

$$x(t) = X_0 \Re\left[\frac{\omega_1^2}{(\omega_0^2 - \omega^2 + i\gamma\omega)} e^{i\omega t}\right],$$

which is the solution of the equation of motion of the mechanical oscillator subject to an external sinusoidal force.

[2]The linearity assures the system response to be proportional to the excitation. In other words, this implies that the system cannot oscillate at frequencies different from the excitation frequency.

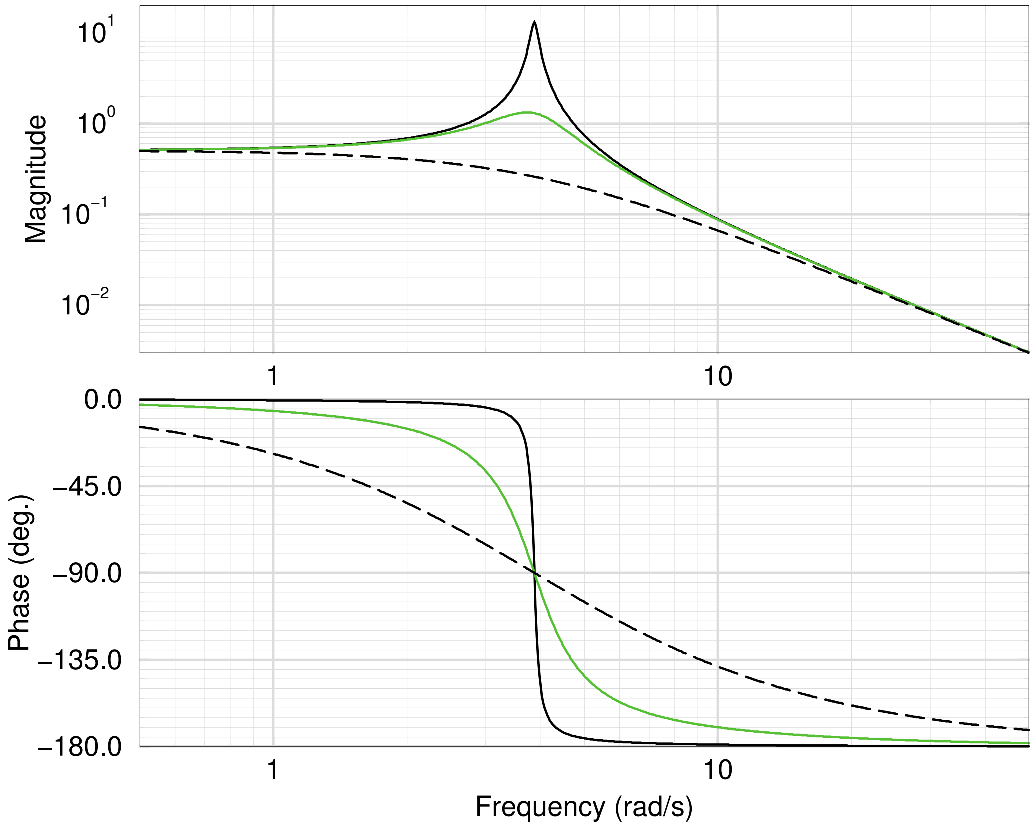


Figure 3.4: Amplitude and phase of the transfer function $H(\omega)$of the mechanical harmonic oscillator for three different values of γ.

3.2.2 Transfer Function of the Mechanical Oscillator

The following quantity

$$H(\omega) = \frac{\text{Bob-displacement}}{\text{Actuator-displacement}} = \frac{X}{X_0},$$

is the *transfer function* or *transmissibility* of the mechanical oscillator. The system response (the dynamics of the bob) is univocally determined once its transfer function is known. For example, the response of the system for a sinusoidal excitation of angular frequency ω is

$$x(t) \quad = \quad X_0 \Re\Big[H(\omega) e^{i\omega t}\Big] = X_0 |H(\omega)| \Re\Big[e^{i(\omega t + \varphi)}\Big] = X_0 |H(\omega)| \cos(\omega t + \varphi).$$

Computing the absolute value and the phase of $H(\omega)$ we obtain

$$|H(\omega)| = \frac{\omega_1^2}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}, \tag{3.5}$$

$$\arg[H(\omega)] = \varphi(\omega) = -\arctan\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right) \tag{3.6}$$

At low frequency, the asymptotic behavior is

$$\omega \ll \omega_0 \quad \Rightarrow \quad \begin{cases} |H(\omega)| & \simeq \frac{\omega_1^2}{\omega_0^2}\,, \\ \varphi(\omega) & \simeq 0\,, \end{cases}$$

and at high frequency

$$\omega \gg \omega_0 \quad \Rightarrow \quad \begin{cases} |H(\omega)| & \simeq \frac{\omega_1^2}{\omega^2}\,, \\ \varphi(\omega) & \simeq -\pi\,. \end{cases}$$

The maximum of $|H(\omega)|$ is for $\omega = \omega_{\max}$, i.e.

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2} \quad \Rightarrow \begin{cases} |H(\omega_{\max})| & = \frac{1}{\gamma}\frac{\omega_1^2}{\omega_\gamma} \\ \varphi(\omega_{\max}) & = -\arctan\sqrt{4\frac{\omega_0^2}{\gamma^2}-2}\,. \end{cases}$$

If γ is much smaller than ω_0 we have

$$\gamma \ll \omega_0 \quad \Rightarrow \quad \varphi(\omega_{\max}) \simeq -\frac{\pi}{2}\,.$$

3.3 Viscous Damping

The viscous damping of the previous mechanical oscillator can be studied considering the system without the two springs and with a constant known force acting on the mass. This constant force can be obtained using the gravity field, i.e. tilting the air bearing guide (see figure 3.5).

In the steady state regime, the constant force F_g will be balanced by the friction force F_v produced by the Eddy currents and the mass will travel at

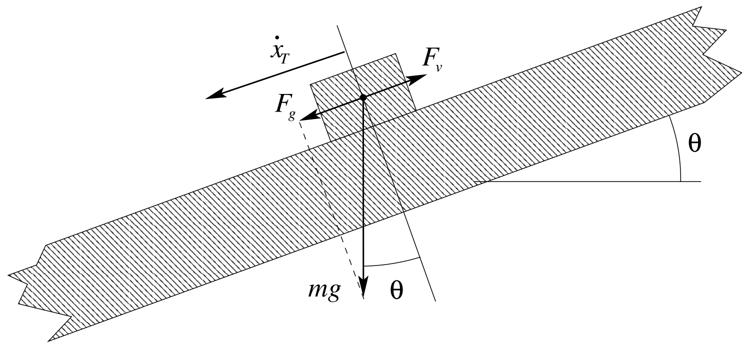


Figure 3.5: Terminal velocity of the mass.

a constant terminal velocity $\dot{x}_T$. Imposing the steady state condition, we have that the magnitude of the two forces

$$F_g = mg\sin\theta, \qquad F_\nu = m\gamma\dot{x}_T,$$

must be equal. This leads to

$$\gamma = \frac{g\sin\theta}{\dot{x}_T},$$

which is the equation that relates the viscous damping coefficient normalized to the mass to the terminal velocity of the mass.

3.4 Effective Mass of a Real Spring

A simple way to take into account the mass M of a spring of length L is to consider the discrete model shown in figure 3.6. The model is made of of N point-like masses μ (representing the distributed spring mass) with coordinates $x_1, x_2, ..., x_N$ connected by N massless springs of rest length l where

$$l = \frac{L}{N}, \qquad \mu = \frac{M}{N}.$$

Then, the total kinetic energy associated with the masses will be

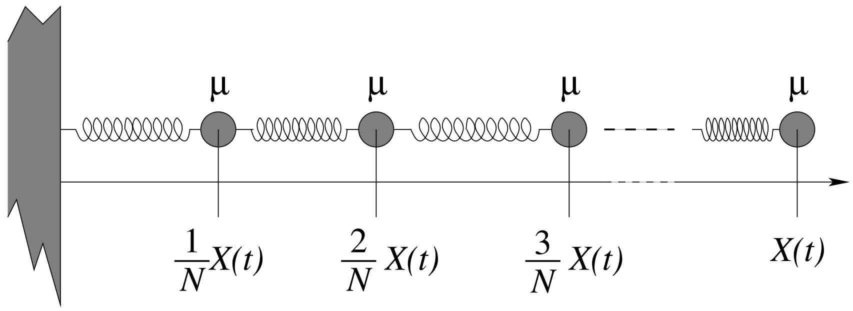


Figure 3.6: Simple discrete model of a spring.

$$T_N = \sum_{n=1}^{N} \frac{1}{2}\mu \dot{x}_n^2 = \frac{1}{2}\frac{M}{N}\sum_{n=1}^{N}\dot{x}_n^2.$$

Supposing that all the N masses move in the same direction, and each ideal spring stretches uniformly by the same amount [3], we can write

$$x_n(t) = \frac{n}{N}X(t), \qquad n = 1, 2, ..., N.$$

Substituting the previous expression into the kinetic energy we obtain

$$T_N = \frac{1}{2}M\dot{X}^2\frac{1}{N^3}\sum_{n=1}^{N} n^2 = \frac{1}{2}M\dot{X}^2\frac{N(N+1)(2N+1)}{6N^3}.$$

Considering that

$$\lim_{N\to\infty}\frac{N(N+1)(2N+1)}{6N^3} = \frac{1}{3},$$

we finally obtain the expression for the kinetic energy of the spring

$$T_\infty = \frac{1}{2}\frac{M}{3}\dot{X}^2.$$

The spring contribution to the kinetic energy is equivalent to a rigid body having a mass equal to 1/3 of the spring mass.

[3] We are not interested on the internal vibrational modes of the real spring in this very simple model.

3.5 Experimental Apparatus

To realize a nearly frictionless mechanical oscillator, the experimental apparatus has a guide, called air trough. The air trough creates a thin "air film" (less than 80μm) upon which the oscillating mass (the glider) is free to move in one dimension.

Two helical springs are attached to the glider. One spring is then connected to the edge of the trough, and the other to a motor through and eccentric pulley to provide a sinusoidal motion.

Eddy current damping is achieved by a permanent magnet placed on the glider.

Positions of the glider and the actuator are measured using optical position sensors connected to a data acquisition board (calibration of the position sensors $\frac{\Delta n}{\Delta s} = 8$count/mm with a resolution of $\Delta n = 1$count and the uncertainty on the time is $\sigma_t = 0.2$ms.).

The slope of the air trough can be set turning a vertical screw placed at one end of the air trough (leveling screw calibration factor $\frac{\Delta \theta}{\Delta s}1$mrad/rotation. Due to local dips and humps, the uncertainty in the angle has been estimated to be $\sigma_{\Delta\theta} = 0.2$mrad).

3.5.1 Care and Use of the Experimental Apparatus

The air trough is particularly delicate because of the the air film thickness. Any scratch or dirt on the glider or on the trough can compromise the use of the experimental apparatus. These are the precautions that need to be taken:

1. TURN THE AIR SUPPLY TO 20PSI BEFORE ANY OPERATION.
2. WITH THE AIR SUPPLY ON, CLEAN THE TROUGH WITHOUT THE GLIDER AND THE SPRINGS WITH PAPER TISSUES MOISTENED WITH ALCOHOL.
3. TO REMOVE THE SPRINGS FROM THE GLIDER OR THE GLIDER ITSELF, LIFT AND HOLD THE GLIDER OUT OF THE AIR TROUGH.
4. DO NOT LET ANY OBJECT FALL DOWN INTO THE TROUGH.
5. REMEBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

3.6 First Laboratory Week

Sections 3.1, 3.3, and 3.5 must be carefully studied before doing the preparatory problems. Section 3.4 is facultative.

3.6.1 Pre-laboratory Problems

1. Considering that for $x(\tau)/x_0 = 1/e \simeq 1/3$, estimate the time constant τ from figure 3.2.

2. Determine the units of $\omega_0^2 = (k_1 + k_2)/m$ and of γ.

3. Supposing that $\omega_0 = 4$ rad/s, calculate for which values of γ, $|\omega_\gamma - \omega_0|/\omega_0 \leq 1\%$.

3.6.2 Procedure

Measure the following physical quantities:

1. Determine the following parameters of the mechanical oscillator by measuring its damped oscillation: the quality factor Q , the time constant τ, the viscous damping coefficient γ, and the resonant frequency ω_γ.

2. Determine the viscous damping γ coefficient by measuring the terminal velocity $\dot{x}_T$.

3. measuring the two spring constant k_1, k_2, and the glider mass m determine ω_0 . Using the previous measurement of γ and this new value of ω_0, calculate the resonant frequency ω_γ.

4. REMEBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

The following measurements are facultative:

- Determination of the energy percentage $\frac{\Delta E(T)}{\Delta E(0)}$ dissipated per cycle T by the mechanical oscillator.

- Measurement of the quality factor Q as a function of the permanent magnet position.

- Calibration of the position sensors.
- Calibration of the leveling screw.

3.7 Second Laboratory Week

Sections 3.2, and 3.5 must be carefully studied before doing the preparatory problems

3.7.1 Pre-laboratory Problems

1. Does the sinusoidal force F necessary to drive the mechanical oscillator at a fixed amplitude x depend on frequency? Prove it.
2. Considering the following parameters for the mechanical oscillator, $m = 0.6\text{kg}$ $\omega_\gamma = 3.8\text{rad/s}$, $\gamma = 0.17\text{s}^{-1}$, compute the force necessary to move the glider by 1mm (static regime).
3. Redo the previous calculation in the case of a sinusoidal force with angular frequency $\omega = \omega_{\text{max}}$ (dynamic regime).
4. Linearize the phase of the transfer function $\varphi(\omega)$ to obtain ω_0, and γ from a linear fit (hint: if $y = ax + b$ then $x = \omega/\tan\varphi$, and $y = \omega^2$).
5. Because of the linearity of the system, the solution of the mechanical oscillator subject to a sinusoidal force and a step response is

$$x(t) \quad = \quad x_0 e^{-\gamma t/2}\cos(\omega_\gamma t + \varphi_0) + |H(\omega)|X_0\cos[\omega t + \varphi(\omega)].$$

   Supposing that for a given frequency $|H(\omega)|X_0 = x_0$ and $\gamma = 0.15\text{s}^{-1}$, calculate the time $\tau*$ necessary for the step response to contribute by 1% on the amplitude of the oscillation.

3.7.2 Procedure

Measure the following physical quantities:

1. Measure and plot it using the appropriate scales, the transfer function $|H(\omega)|$, $\arg(H(\omega))$ of the forced mechanical oscillator.

2. Determine from the magnitude of the transfer function γ, ω_0, and the maximum of $|H(\omega)|$.

3. Determine ω_γ using the previous measurements.

4. Determine the angular frequency ω_0 and viscous damping coefficient γ from the transfer function phase $\arg(H(\omega))$.

5. Compare all the new measurements of γ with those ones of the previous week.

6. REMEBER TO CLOSE THE AIR SUPPLY OUTPUT ONCE FINISHED.

Chapter 4

The Inverted Pendulum (IP)

4.1 Introduction

The inverted pendulum (shortly IP) is a mechanical harmonic oscillator whose peculiarity is to make possible to obtain a very low resonant frequency in the horizontal direction. In fact, using the restoring torque of a flex joint, to balance the torque due to the gravity force (see figure 4.1), an inverted pendulum with a leg of 1m can be tuned to reach a resonant frequency below 100mHz. A similar device using a simple pendulum with the same characteristics would be quite impractical.

4.2 A Simple Model

To understand the dynamics of the inverted pendulum, we will consider a simplified model with the following definitions and approximations (see figure 4.1):

- a massless leg of length l,
- a mass M all concentrated in one point,
- dynamics completely described by one angular coordinate, i.e. the angle θ between the vertical axis and leg axis,

- flex joint length negligible making a force described by the Hooke's law with an equivalent of a spring constant k

$$F = -kx = -kl\theta.$$

- no energy dissipation mechanisms.

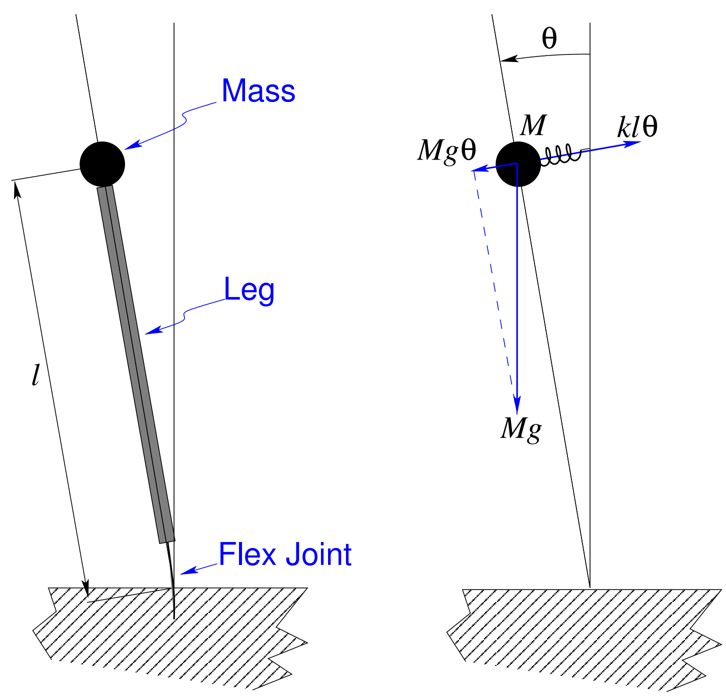


Figure 4.1: Simple model of the inverted pendulum.

4.2.1 The Equation of Motion

Considering this simplified model of the inverted pendulum in the harmonic oscillation regime ($\theta \ll 1$), the equation of motion is

$$Ml\ddot{\theta} = -kl\theta + Mg\theta, \qquad \theta \ll 1.$$

Rearranging the previous equation, we get

$$\ddot{\theta} + \omega_0^2\theta = 0, \qquad \omega_0^2 = \frac{k}{M} - \frac{g}{l}, \tag{4.1}$$

which is the linear differential equation of an harmonic oscillator with *angular resonant frequency* ω_0, whose general solution is

$$\theta(t) = \theta_0 \cos(\omega_0 t + \varphi_0).$$

As usual, the constants θ_0 and φ_0 depend on the initial conditions at $t = 0$.

4.3 A Better Model of the Inverted Pendulum

One simple improvement of the previous inverted pendulum model is to consider the mass m and the moment of inertia I about the axis $O\hat{x}$ of the leg. Using the second law of dynamics we have[1]

$$(I + Ml^2)\ddot{\theta} = -kl^2\theta + Mgl\theta + mgd\theta. \tag{4.2}$$

where d is the leg´s center of mass distance from the pivot point O. The solution is the same as the previous equation (4.1) with the difference that the angular resonant frequency is now

$$\omega_1^2 = \frac{kl^2 - (Ml + md)g}{I + Ml^2}. \tag{4.3}$$

In case of a cylindrical tube of negligible wall thickness, outer radius r length H and mass m, the leg´s moment of inertia about the pivot point O is

$$I = I_0 + md^2, \qquad I_0 = \frac{1}{2}m\left(\frac{H^2}{6} + r^2\right)^2.$$

[1]Because we are still considering the mass M a point like mass, its moment of inertia is its mass M times the square of the arm lever l.

4.4 Energy Dissipation Mechanisms

Another step to improve our model is to include the mechanical energy dissipation which is always present in any physical system. Essentially, There are two major dissipation mechanisms, the so called structural, and viscous damping, and their net effect is to damp the motion.

The viscous damping can be described as friction between the mechanical system and the viscous medium, (the air in our case) where the system moves through. For an inverted pendulum in air, it becomes dominant with respect to other dissipation mechanisms at velocities higher than the typical resonances.

For structural damping case, the mechanical energy loss is due to the non perfectly elastic materials. In other words, when those materials are are bent, compressed or stretched the energy stored into them is dissipated as heat, and it is not returned to the mechanical system[2].

Experimentally, the structural damping produces a sinusoidal motion, which decays exponentially. In this case, a way "ad hoc" of taking into account the structural damping is to rewrite the equation of motion with a complex spring constant k

$$k \quad \Rightarrow \quad k(1 + i\phi) \, ,$$

where ϕ is said to be the *loss angle*.

Equation (4.2) becomes[3]

$$(I + Ml^2)\ddot{\theta} \quad + [k(1 + i\phi)l^2 - Mgl - mgd]\,\theta = \quad 0\,. \tag{4.4}$$

Defining the following quantity

$$\omega_2^2 = \frac{kl^2}{I + Ml^2},$$

and rearranging equation 4.4, we get

$$\ddot{\theta} + \left(\omega_1^2 + i\phi\omega_2^2\right)\theta = 0,$$

[2]"Thermodynamically speaking" it means that there are some irreversible processes.

[3]It is important to notice that because of the presence of the imaginary term equation (4.4) does not have a physical meaning anymore.

and the solution of this equation becomes

$$\theta(t) = \theta_0 e^{-\phi\omega_2 t} \cos(\omega_1 t + \varphi_0), \tag{4.5}$$

which is an exponentially decaying sinusoid with the angular resonant frequency independent from the dissipation mechanism.

The inverse of the loss angle $Q = 1/\phi$ is the *quality factor* or *merit factor of the oscillator*. The $i\phi$ term introduced in the equation of motion is indeed just an artifact to obtain the right solution.

4.5 Forced Oscillator (Sinusoidal Excitation Response)

Introducing a new coordinate system, the displacement of the mass M along the x axis equation (4.5) becomes

$$\ddot{x} + \left(\omega_0^2 + i\phi\omega_2^2\right) x = 0, \qquad x = \theta l. \tag{4.6}$$

If we excite the IP clamping point with a force, which produces a displacement x_0, then the new position x of the mass M is

$$x \Rightarrow x - x_0.$$

Using the previous transformations, equation (4.6) becomes

$$\ddot{x} + \left(\omega_0^2 + i\phi\omega_2^2\right)(x - x_0) = 0\ .$$

Collecting x_0 on the right-hand side we get

$$\ddot{x} + (\omega_0^2 + i\phi\omega_2^2)x = (\omega_0^2 + i\phi\omega_2^2)x_0.$$

For a sinusoidal excitation $x_0(t) = X_0 \Re\left[e^{i\omega t}\right]$[4], the solution of the previous equation must be in the form $x(t) = \Re\left[X e^{i\omega t}\right]$, with X as a complex unknown amplitude independent from time t. Substituting these functions into the previous equation, we get

$$-\omega X + (\omega_0^2 + i\phi\omega_2^2)X = (\omega_0^2 + i\phi\omega_2^2)X_0.$$

[4] $\Re[...]$ means the real part of what is in the square brackets.

and finally

$$X = \frac{\omega_0^2 + i\phi\omega_2^2}{\omega_0^2 - \omega^2 + i\phi\omega_2^2} X_0.$$

The solution of the inverted pendulum subject to a sinusoidal excitation with angular frequency ω on the clamping point is indeed

$$x(\omega,\, t) = \Re\left[\frac{\omega_0^2 + i\phi\omega_2^2}{\omega_0^2 - \omega^2 + i\phi\omega_2^2} X_0 e^{i\omega t}\right].$$

4.5.1 The Inverted Pendulum Transfer Function

The following expression

$$H(\omega) = \frac{X}{X_0},$$

is defined the transfer function of the system (or the transmissibility for mechanical systems). The knowledge of the transfer function determines univocally the behavior of a linear system. Knowing the amplitude of the input X_0 for each frequency we can compute the output by multiplying it by the function $H(\omega)$.

For the inverted pendulum we will have

$$\begin{aligned} |H(\omega)|^2 &= \frac{\omega_0^4 + \phi^2\omega_2^4}{(\omega_0^2 - \omega^2)^2 + \phi^2\omega_2^4} \\ \arg[H(\omega)] &= \varphi(\omega) = -\arctan\left(\frac{\phi\omega_2^2\omega^2}{\omega_0^4 - \omega_0^2\omega^2 + \phi^2\omega_2^4}\right) \end{aligned}$$

The phase qualitative behavior can be understood considering $\phi \ll 1$, which implies that $\phi^2 \simeq 0$, Computing the phase for the following angular frequency values

$$\begin{aligned} \omega \ll \omega_0 \quad &\Rightarrow \quad \varphi(\omega) \simeq 0, \\ \omega = \omega_0 \quad &\Rightarrow \quad \varphi(\omega) \simeq -90^o, \\ \omega \gg \omega_0 \quad &\Rightarrow \quad \varphi(\omega) \simeq -180^o \end{aligned}$$

For the magnitude it is easy to see that for

$$\omega \ll \omega_0, \quad \Rightarrow \quad |H(\omega)| = 1,$$

$$\omega = \omega_0, \quad \Rightarrow \quad |H(\omega_0)|^2 = 1 + \left(\frac{\omega_0^2}{\omega_2^2\phi}\right)^2,$$
$$\omega \gg \omega_0, \quad \Rightarrow \quad |H(\omega)| \simeq \frac{\omega_0^2 + \phi\omega_2^4}{\omega^2}$$

With the energy dissipation the qualitative behavior changes essentially around the resonance ω_0, where $|H(\omega)|$ does not diverge anymore. Figure ?? shows the transfer function in a logarithmic scale for the angular frequency and for the amplitude$|H(\omega)|$.

4.6 The Inverted Pendulum Test Bench

Figure 4.6 shows a schematic of the inverted pendulum test bench. The device is essentially made of 6 mechanical functional parts:

1. the **base**, to support the mechanical system, with three leveling thumb screws,

2. the **actuator,** a platform suspended to the a little table by four shims 0.3mm thick, 4.76mm wide, and 75mm long, clamped to the platform and to the table,

3. the aluminum **leg**, essentially a light hollow cylinder,

4. the **flex joint,** 16mm long and 2mm,[5]

5. the **threaded rod,** to symmetrically load the IP.

6. **A stage** on the side of the inverted pendulum leg, to mount the IP travel limiter, an support the velocity sensor.

An **electro-magnetic actuator** is mounted between the base and the actuator platform to excite the IP. One velocity sensors is placed close as possible to the IP load to measure the velocity of the IP's top. Another velocity sensor mounted on the base measures the actuator platform's velocity.

[5]The flex joint is machined from a steel square beam

4.7 First Laboratory Week

4.7.1 Preparatory Laboratory Problems

1. Rewrite equation (4.3) considering the parameter **angular stiffness** $\kappa = k/l$. What are the units of κ?

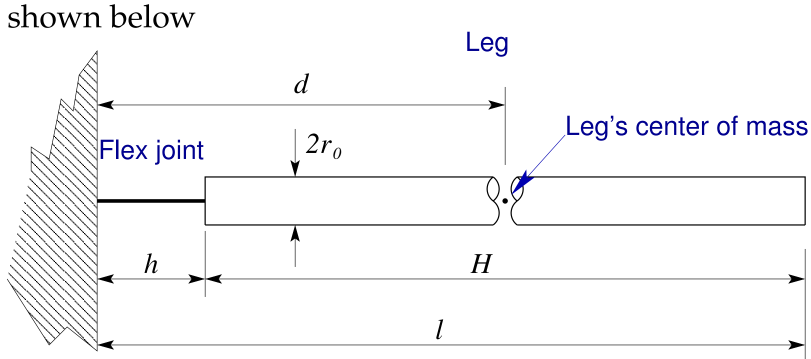
2. Determine the length of an IP leg with a load of $M = 0.300$kg, flex joint flexural angular stiffness $\kappa = 10$N/m^2, resonant frequency $\nu_0 = 100$mHz, and negligible leg´s mass m. Compute the length of the equivalent simple pendulum (i.e. having the same resonant frequency).

3. Compute the force needed to hold the load´s M center of mass 5mm far from the equilibrium position for the IP of problem 2.

4. Calculate the mass variation ΔM needed to change the frequency from 100mHz to 10mHz for the IP of problem 2.

5. Consider the IP of problem 2 which is ringing down after a step excitation. Supposing that the loss angle is $\phi = 10^{-2}$, and the leg´s moment of inertia is $I = 3.2 \cdot 10^{-3}$kgm^2, compute the time necessary to have the IP oscillation amplitude 1% of the initial value x_0.

6. Rewrite the moment of inertia expression I using the parameters h, and l as shown below



Compute the leg's moment of inertia I with $m = 25$mg, $l = 650$mm, $h = 28$mm , and $r_0 = 5$mm, and the systematic error on I if we neglect the leg´s radius.

4.7.2 Care and Use of the Experimental Apparatus

The flexural joint is particularly delicate because of its small stiffness. Large angles can make the joint working in the plastic regime causing an irreversible damage.

These are the main precautions that need to be taken:

1. NEVER LET THE IP OSCILLATE WITHOUT THE TRAVEL LIMITER.
2. DO NOT LET THE IP LEG FALL.
3. TO DISASSEMBLE THE IP LEG, LOCK THE ACTUATOR PLATFORM, REMOVE THE LOAD FROM THE LOAD PLATE AND INSERT THE PROPER WRENCH KEY IN THE LOWER PART OF THE FLEXURAL JOINT TO AVOID THAT A TORQUE IS DIRECTLY APPLIED TO THE JOINT.

4.7.3 Procedure

Read this text down to the end before starting your laboratory work, and **remember to specify in your report which set-up you used.**

All the measurement must be performed with the **actuator platform locked with the thumb screws to the IP base**.

The practical difficulty of this experiment is to achieve the IP equilibrium position for each different load. The lower the resonant frequency the more sensitive is the IP to a load displacement.

To keep the IP in equilibrium, coarsely level the leg's top using the bubble level, and acting on the three leveling screws.

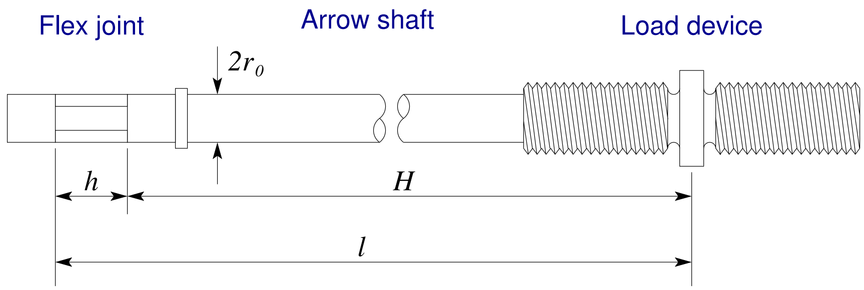
The previous procedure assures that the IP equilibrium position is not too far from the vertical axis going through the center of the travel limiter, but it doesn't assure that at the equilibrium the IP leg axis will coincide to this axis. Moreover, if we add a new weight on the load plate the equilibrium position will change.

When a new mass placed on the load plate makes the IP touching the travel limiter re-level the IP acting on the leveling screws. Alternatively, walk a small weight around on the load to bring the IP back to the equilibrium.

When the resonant frequency is below 200mHz, the hysteresis becomes not negligible anymore, and strong and sudden perturbation can bring the IP far from the previous equilibrium position.

4.7.4 Inverted Pendulum Leg

Leg´s dimension are sketched the figure below



The load device is designed to place the load symmetrically respect the point A keeping the length of the leg l constant.

Use the spare leg, the dummy flex joint, and the load device to do the measurements of l, H, and h.

Resonant Frequency vs Load Measurement

- Experimentally, study the variation of the IP resonant frequency ν_0 as a function of the load M placed on top of the IP. The measurement of ν_0 can be indirectly done measuring severals oscillation periods T and using the provide data acquisition system.

- Determine the flex joint angular stiffness $\kappa = k/l$ fitting equation 4.3.

Inverted Pendulum Loss Angle

- Using the IP ring-down amplitude envelope, determine the loss angle ϕ of the IP with no load, with an intermediate load M, and the with a load as close as possible to maximum load you placed on the IP.

To record the amplitude of the ring-down of the amplitude, use the velocity sensor placed closed to the load and the available data acquisition system.

4.8 Second Laboratory Week

4.8.1 Preparatory Laboratory Problems

1. Demonstrate that $H(\omega) = \frac{\dot{X}}{\dot{X}_0}$, i.e. we can measure the transmissibility of the IP measuring the velocity of the Load $\dot{X}$ and the velocity of the actuator platform $\dot{X}_0$.

2. Consider IP of problem 2 of the previous week with loss angle $\phi = 10^{-3}$, and resonant frequency $\nu_0 = 100$mHz. Driving the load´s center of mass at the resonant frequency ν_0, compute the force needed to make the IP load´s center of mass oscillate with an amplitude of 5mm.

3. Repeat the same calculation of problem 2 supposing no dissipation $\phi = 0$.

4. Repeat the same calculation of problem 2 for a driving frequency $\nu = 100$Hz.

5. Supposing that the actuator force F is proportional to the current i , i.e $F = \alpha i$, $\alpha = 0.25$N/A, compute the current necessary to move the IP load of problem 2 by 5mm at the resonance ν_0 and at 100Hz.

4.8.2 Procedure

Read this text up to the end before staring your laboratory work, and remember to specify in your report which set-up you used.

All the measurement must be performed with the actuator platform **unlocked**.

IP Transmissibility

- Measure the transmissibility of the IP using the two velocity sensors, and the same load M_0 used before.
- Try to repeat the previous measurement with the load M_{max}.
- Compare the two set of measurement of the loss angle at the resonant frequency.

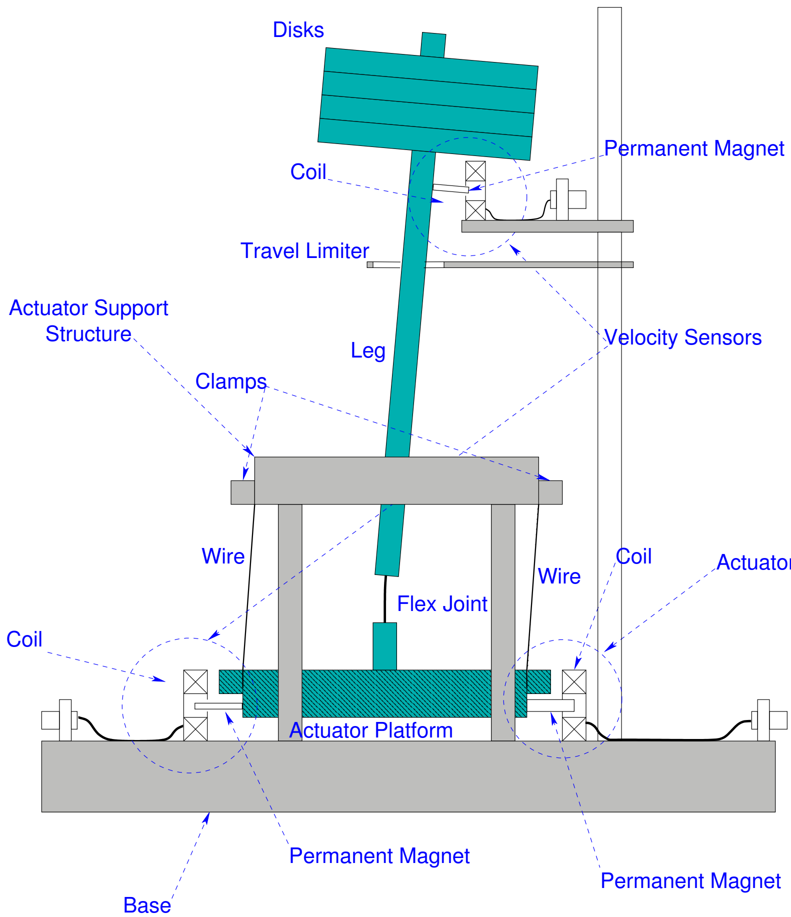


Figure 4.2: Inverted pendulum test bench .

Chapter 5

Direct Current Network Theory

5.1 Electronic Networks

An *electronic network* or circuit is a set of electronic components/devices connected together to modify and transmit/transfer energy/information. This information is generally called *electric signal* or simply signal. To graphically represent a network, we use a set of coded symbols with terminals for devices and lines for connections. These lines propagate the signal among the devices without changing it. Devices change the propagation of the electric signals instead.

Quantities defining this propagation are the voltages V across the devices and currents I flowing through them.

Solving an electronic network means determine the currents or the voltages on each point of it.

To make the understanding of network basic theorems easier, some more or less intuitive definitions must be stated.

5.1.1 Network Definitions

A *network node* is a point where more than two network lines connect.

A *network loop, or mesh,* is any closed network line. To determine a mesh it is sufficient to start from any point of the circuit and come back running through the network to the same point without passing through a same point.

Figure 5.1 shows a generic portion of a network with 4 visible nodes,

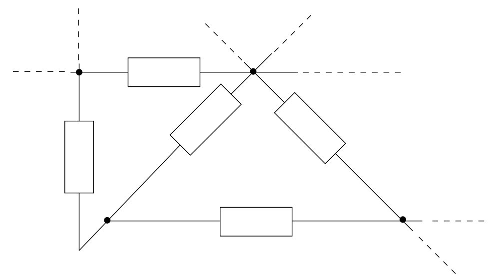


Figure 5.1: Generic representation of a network

and 3 visible meshes. The empty boxes are the electronic components of the network. The electronic components are points of the network.

5.1.2 Series and Parallel

Let's consider the two different connection topologies shown in figure 5.2, the parallel and the series connections.

A set of components is said to be in series if the current flowing through them and anywhere in the circuit is the same .

A set of components is said to be in parallel if the voltage difference between them is the same.

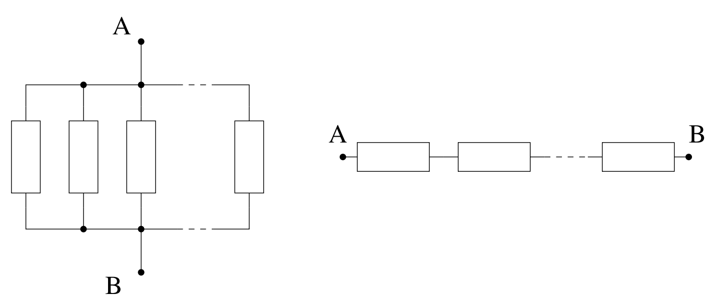


Figure 5.2: Considering the points A, and B, the components of the left circuit are in parallel, and those ones in the right circuit are in series.

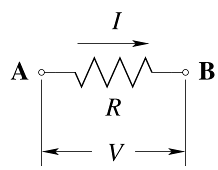


Figure 5.3: Resistor symbol

5.1.3 Active and Passive Components

Circuit components can be divided into two categories: active and passive components. Active components are those devices that feed energy into the network. Voltage and current sources are active components. Amplifiers are also considered active components.

Passive components are those components that do not feed energy to the network. Resistors, capacitors, inductors are typical passive components.

In general, both active and passive components dissipate energy.

5.2 Kirchhoff's Laws

In this section the two Kirchhoff's laws, which are fundamental for the solution of an electronic circuit are stated here.

Kirchhoff's Voltage Law (KVL): The algebraic sum of the voltage difference v_k around a mesh must be equal to zero at all times, i.e.

$$\sum_k v(t) = 0$$

Kirchhoff's Current Law(KCL): The algebraic sum of the currents i_k entering and leaving a node must be equal to zero at all times, i.e.

$$\sum_k i_k(t) = 0.$$

5.3 Resistors (Ohm's Law)

It is experimentally known that if we apply a voltage V across a metal (a conductor), we will measure an electric current I flowing through it. V results to be proportional to I by a constant R, named electric *resistance* of the conductor. In other terms we have

$$\frac{V}{I} = R, \tag{5.1}$$

which is called *Ohm's law*. The unit for the resistance is the "Ohm" whose symbol is the Greek letter Ω. It follows from Ohm's law that $[\Omega] = $ [V/A]="Volt per Ampere".

In a metallic conductor, R is proportional to the conductor length l and inversely proportional to the cross-section s, i.e.

$$R = \rho\frac{l}{s}.$$

ρ is the conductor *resistivity* and it depends on the metal and its impurities. A device which follows Ohm's Law is said to be a *resistor* and its symbol is shown in figure 5.3.

The power P dissipated by the resistor with resistance R is

$$P = VI, \qquad \Rightarrow \qquad P = \frac{V^2}{R} = RI^2.$$

Let's define some properties of the resistance of conductors.

5.3.1 Resistors in Series

The resistance R_{tot} of a set of resistor $R_{1,}R_2, \ldots, R_n$, connected in series (see figure 5.4), is equal to the sum of the resistances

$$R_{tot} = \sum_{k=1}^{n} R_k.$$

The previous formula is easy to demonstrate. Connecting the resistor series to a voltage source V, and applying the (5.1) to each resistors we have

$$V_1 = R_1 I, \quad V_2 = R_2 I, ... \quad V_n = R_n I,$$

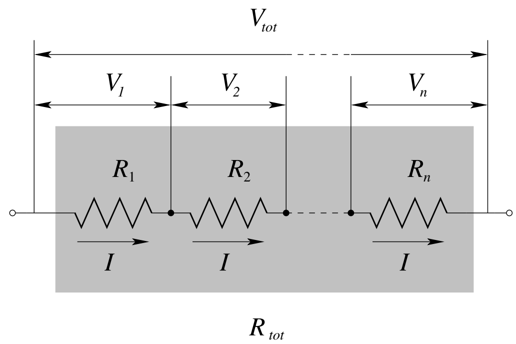


Figure 5.4: Resistor in series

where I is the current flowing through each resistor. Because of the KVL, the voltage difference V must be the sum of all the voltages differences, i.e.

$$V = \sum_{k=1}^{n} R_k I = \left(\sum_{k=1}^{n} R_k \right) I.$$

5.3.2 Resistors in Parallel

The inverse of the resistance R_{tot} of a set of resistor $R_{1,}R_2, \ldots, R_n$, connected in parallel (see figure 5.5), is equal to the sum of the inverse of the resistances

$$\frac{1}{R_{tot}} = \sum_{k=1}^{n} \frac{1}{R_k}.$$

This law can be easily derived using (5.1) and the definition of components in parallel.

5.4 Capacitors

Let's study another typical component of an electronic circuit, the capacitor. A capacitor is a system of two conductors which goes under full induction when a voltage difference is applied to the conductors. Each conductor will be charged with the same amount of charge Q with opposite

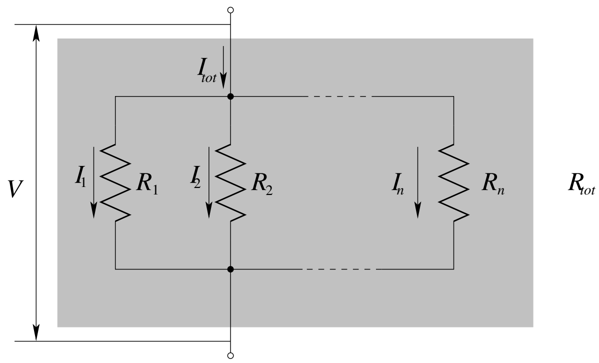


Figure 5.5: Resistors in parallel.

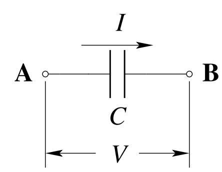


Figure 5.6: Capacitor symbol.

sign. The ratio between the voltage difference V between the two conductors and the charge Q

$$C = \frac{Q}{V} \tag{5.2}$$

is constant and is said to be the capacitance C of the capacitor. The capacitance depends on the geometry of the conductors and on the interposed dielectric. The units for the capacitance is the "Faraday' whose symbol is the letter F. It follows that [F] $=$ [C/V]="Coulomb per Volt".

5.4.1 Capacitors in Parallel

A parallel of capacitors $C_1, C_2, ..., C_n$ is a capacitor whose capacitance C_{tot} is the sum of all the capacitances, i.e.

$$C_{tot} = \sum_{k=1}^{n} C_k. \tag{5.3}$$

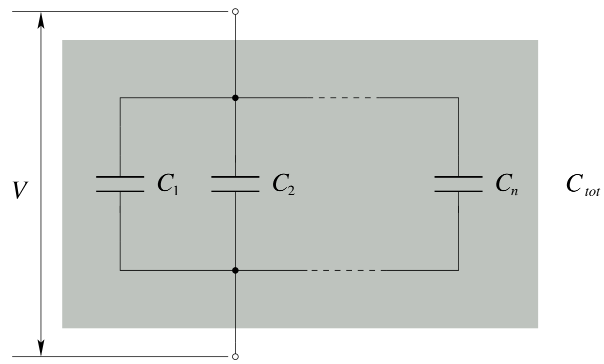


Figure 5.7: Capacitors in parallel.

The previous formula is easy to demonstrate. In fact, the total induced charge Q_{tot} on the capacitors side at the same potential is equal to the sum of all charges of those sides, i.e.

$$Q_{tot} = \sum_{k=1}^{n} Q_k.$$

Considering that the voltage difference V across each capacitor must be same, and dividing by V we obtain the (5.3).

5.4.2 Capacitors in Series

A series of capacitors $C_1, C_2, ..., C_n$is a capacitor with capacitance C_{tot} satisfying the following equation

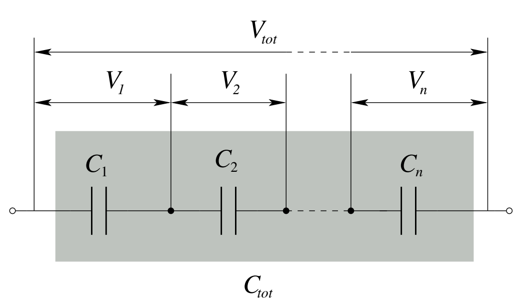


Figure 5.8: Capacitors in series.

$$\frac{1}{C_{tot}} = \sum_{k=1}^{n} \frac{1}{C_k}.$$

The demonstration of the previous equation is left as exercise (hint: in this case the induced charges are equal and not the voltage differences across the capacitors).

5.5 Ideal and Real Sources

5.5.1 Ideal Voltage Source

An ideal voltage source is a source able to deliver a given Voltage difference V_s between its leads independently of the load R attached to it (see figure 5.9). It follows from Ohm's law that a voltage source is able to produce any current I to keep constant the voltage difference V_s across the load R. The symbol for the ideal voltage source is shown in figure 5.9.

Quite often, a real voltage source exhibits a linear dependency on the resistive load R. It can be represented using an ideal voltage source V_s in series with a resistor R_s called input resistance of the source. Applying Ohm's law, it can be easily shown that the voltage and current through the load R are

$$V = \frac{R}{R+R_s} V_s, \qquad I = \frac{V_s}{R+R_s}.$$

If we assume

$$R \gg R_s, \quad \Rightarrow \quad V \simeq V_s, \quad I \simeq \frac{V_s}{R}.$$

Under the previous condition, the real voltage source approximates the ideal case.

5.5.2 Ideal Current Source

An ideal current source is a source able to deliver a given current I_s which does not depend on the load R attached to it (see figure 5.10). It follows from Ohm's law that an ideal current source is able to produce any voltage

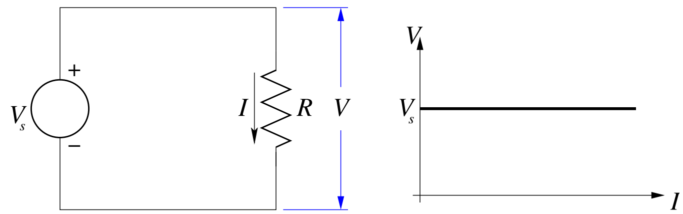


Figure 5.9: Ideal voltage source.

difference V across the load R to keep I_s constant. The symbol for the ideal current source is shown in figure 5.10.

A real current source exhibits a dependency on the resistive load R, which can be represented using an ideal current source I_s in parallel with a resistor R_s. Applying Ohm's law and the KCL, it can be easily shown that the voltage and current through the load R are

$$I = \frac{R_s}{R + R_s} I_s, \qquad V = \frac{R_s R}{R_s + R} I_s.$$

If we suppose

$$R_s \gg R, \quad \Rightarrow \quad I \simeq I_s, \quad V \simeq R I_s \gg 0$$

Under the previous condition, the real current source approximates the ideal case.

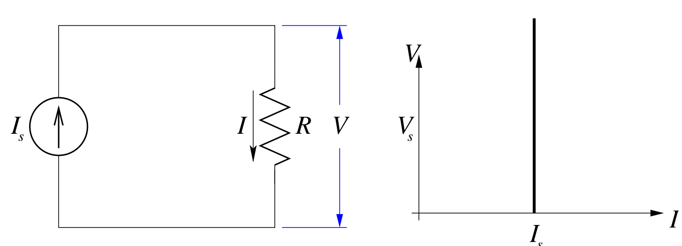


Figure 5.10: Ideal current source.

5.6 The Semiconductor Junction (Diode)

The *semiconductor junction* or *semiconductor diode* is a device which presents a non-linear behavior due to its peculiar conduction mechanism substantially different from the conduction in a metal.

If I_D and V_D are the current and the voltage difference across the junction, we will have

$$I_D(V_D) = I_0(e^{\frac{-qV_D}{\eta k_B T}} - 1), \tag{5.4}$$

where $k_B = 1.3807 \cdot 10^{-23}$J/K is the Boltzmann constant, T the absolute temperature, $q = -1.60219 \cdot 10^{-19}$C the electron charge, and η a dimensionless parameter, which depends on the diode type. I_0 is the reverse saturation current.

Instead of following Ohm's law, the semiconductor junction follows an exponential curve (the diode I-V Characteristic). Deviations from this law are negligible depending on the current magnitude and the diode characteristics.

Figure 5.11 shows the standard symbol for a semiconductor diode and the I-V characteristic. The break-down voltage V_b reported in the same figure is the reverse voltage which essentially shorts circuit the junction. This behavior is not accounted by the equation (5.4).

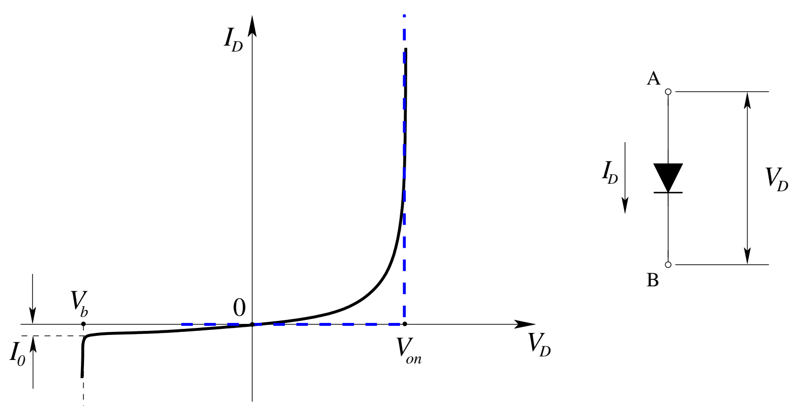


Figure 5.11: Diode characteristic (continuous curve), simplified diode characteristic (dashed curve), and diode standard symbol.

A simplified model of the junction diode is that one of a perfect switch, i.e.

$$I_D(V) = \begin{cases} \infty & V \geq V_{on} \\ 0 & V < V_{on} \end{cases},$$

where V_{on} is the diode *turn-on voltage*, which depends on the junction type and on the current magnitude. For current up to $I_D \sim 100$mA, silicon diodes have $V_{on} \simeq 0.6$V, and germanium diodes have $V_{on} \simeq 0.3$V .

For voltages greater than V_{on}, the diode is a short circuit (current is not limited by the diode) and is said to be *forward biased*. For smaller values it is an open circuit (current across the diode is zero) and is *reverse biased*.

5.7 Equivalent Networks

Quite often, the analysis of a network becomes easier by replacing part of it with an equivalent and simpler network or dividing it into simpler sub-networks. For example, the voltage divider is an equation easy to remember that allows to divide a complex circuit in two parts simplifying the search of the solution. Thévenin and Norton theorems, give us two methods to calculate equivalent simple circuits, which behave like the original circuit seen from two points of it. These techniques briefly explained in this section, will be used also in some of the next experiments.

5.7.1 Voltage Divider

The voltage divider equation is applicable every time we have a circuit which can be re-conducted in a series of two simple or complex components. The simplest case is the one shown in figure 5.12. Applying Ohm's law, we have

$$\begin{aligned} V_{Tot} &= (R_1 + R_2)I \\ V_2 &= R_2 I \end{aligned}$$

and indeed

$$V_2 = \frac{R_2}{R_1 + R_2} V_{Tot},$$

which is the equation of the voltage divider. The equation still holds if we replace the resistance seen from the points A, B for $R_1$and B,C for R_2.

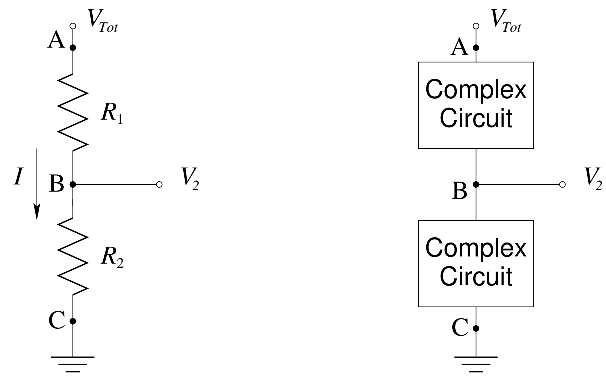


Figure 5.12: Simplest and complex voltage divider circuit.

5.7.2 Thévenin Theorem

Thévenin theorem allows to find an equivalent circuit of a network seen from two points **A** and **B** using the series of an ideal voltage source of voltage V_{Th} and a resistor with resistance R_{Th}.

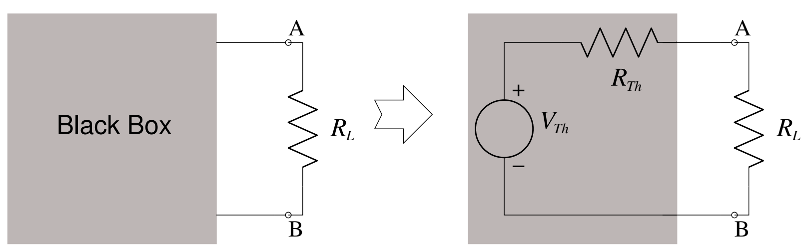


Figure 5.13: Thévenin equivalent circuit illustration.

The equivalence means that if we place a load R_L between **A** and **B** in the original circuit (see figure 5.13) and measure the voltage V_L and the current I_L across the load, we will obtain exactly the same V_L and I_L if $R_L$is placed in the equivalent circuit. This must be true for any load we connect to the points **A** and **B** .

The previous statement and the linearity of the circuit can be used to find V_{Th} and R_{Th}. In fact, if we consider $R_L = \infty$ (open circuit, OC), we will have

$$V_{Th} = V_{OC} \ .$$

The Voltage V_{Th} is just the voltage difference between the two leads **A** and **B**.

For $R_L = 0$ (short circuit, SC) we must have

$$I_{SC} = \frac{V_{Th}}{R_{Th}} = \frac{V_{OC}}{R_{Th}} \, .$$

and therefore

$$R_{Th} = \frac{V_{OC}}{I_{SC}} \, .$$

The last expression says that the Thévenin resistance is the resistance seen from the points **A** and **B** of the original circuit.

If the circuit is known the Thévenin parameter can be calculated in the case of the terminals **A** and **B** open. In fact, V_{Th} is just the voltage across **A** and **B** of a known circuit. Replacing the ideal voltage sources with short circuits (their resistance is zero) and ideal current sources with open circuits (their resistance is infinite) we can calculate the resistance R_{Th} seen from terminals **A** and **B**.

Example:

We want to find the Thévenin circuit of the network enclosed into the gray rectangle of figure 5.14. To find R_{Th}, and V_{Th} we have to disconnect the circuit in the points **A** and **B.** In this case voltage difference between these two points, thanks to the voltage divider equation, is

$$V_{Th} = \frac{R_1}{R_1 + R_2} V_s.$$

Short circuiting V_s we will have R_1 in parallel with R_2. The Thévenin resistance R_{Th} will be indeed

$$R_{Th} = \frac{R_1 R_2}{R_1 + R_2}.$$

Considering the previous results, we can finally state Thévenin theorem as follows:

Any circuit seen from two points can be replaced by a series of an ideal voltage source of voltage V_{Th} and a resistor of resistance R_{Th}. V_{Th} is the voltage difference between the two point of the original circuit. R_{Th} is the resistance seen from these *two points, short-circuiting all the ideal voltages generators and open-circuiting all the ideal current generators.*

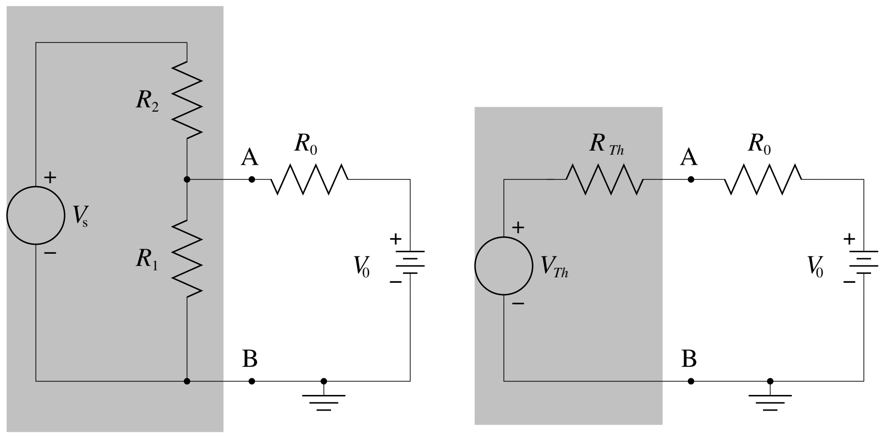


Figure 5.14: Thévenin equivalent circuit example

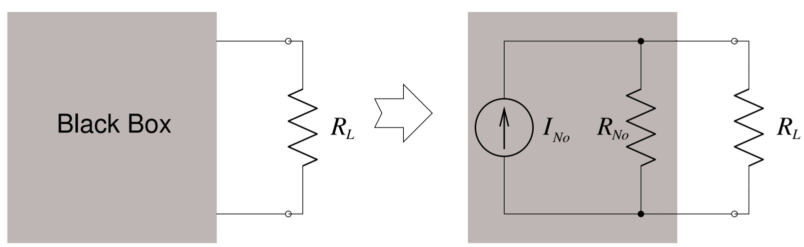


Figure 5.15: Norton equivalent circuit illustration.

5.7.3 Norton Theorem

*Any kind of active network seen from two points **A** and **B** can by replaced by an ideal current generator I_{No} in parallel with a resistance R_{No}. The current I_{No} correspond to the short-circuit current of the two points **A** and **B**. The Resistance R_{No} is the Thévenin resistance $R_{No} = R_{Th}$.*

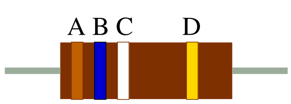
The proof of this theorem is left as exercise.

5.8 Resistor Color Code

Nominal values of resistances are coded using colors bands around the resistors (see figure below). The bands identify digits and the exponent in base ten for the resistance value and the tolerance as explained in the following table:

Band Number	1	2	3	4 (Tolerance band)	5
3 Bands	Digit	Digit	Exponent	Always 20%	
4 Bands	Digit	Digit	Exponent	Tolerance	
5 Bands	Digit	Digit	Exponent	Tolerance	Tolerance after 1000 hours

3 Band resistors have no band for the tolerance because it is assumed to be 20% of the nominal values.The fifth band is not an industry standard, but quite often it means the tolerance after 1000 hours of continuous use.



$$R = AB \cdot 10^C, \quad \Delta R = R \cdot D$$

The bands are counted from left right. The following table reports the coding of the values using colors and a mnemonic sentence to remember the color code table.

	Color	Exponent	Tolerance(%)	Tolerance (%) 5th Band
Big	Black	0	20	
Bart	Brown	1	1	1%
Rides	Red	2	2	0.1%
Over	Orange	3		0.01%
Your	Yellow	4		0.001%
Grave	Green	5		
Blasting	Blue	6		
Violent	Violet	7		
Guns	Gray	8		
Wildly.	White	9		
Go	Gold	-1	5	
Shoot (him?)	Silver	-2	10	

For example, the nominal resistance of a 4 band resistor having the sequence brown, black, orange and gold is

$$\begin{array}{r} R_{nom.} = 10\text{k}\Omega \\ \Delta R_{nom.} = 5\%10\text{k}\Omega \end{array} \quad \Rightarrow \quad R_{nom.} \quad = (10.0 \pm 0.5)\text{k}\Omega$$

Resistor size (volume) is related to the power dissipation capability. Typical used values are 1/4W 1/2W, 1W.

5.9 First Laboratory Week

Sections 1, 2, 3, 5, and 7 of this chapter are required to complete the first laboratory week. A particular attention deserves the Thévenin equivalent circuit, which is the main topic of the experiment.

To experimentally study the basics of electronic networks, which is the scope of this laboratory week, we will use the following instruments:

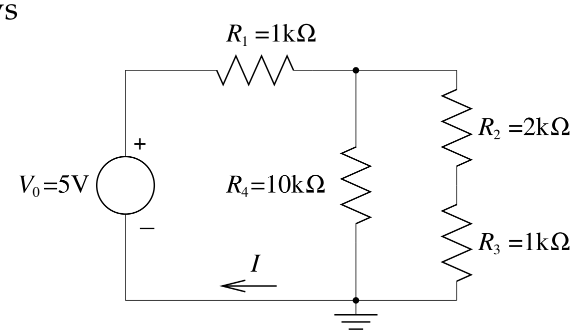
- a digital multimeter (DMM) to measure voltage differences, currents, resistances
- a data acquisition system (DAQ),
- a voltage source.

Whenever you work with electronic circuits as a beginner (all ph3 students are considered beginners it doesn't matter which personal skills they already have), some extra precautions must be taken to avoid injuries. These are the main ones:

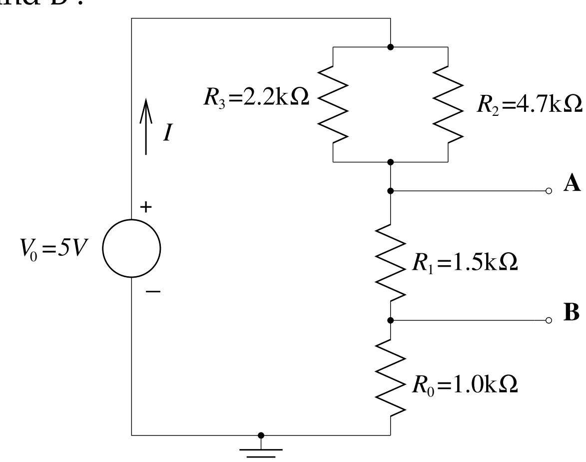
- NEVER CONNECT INSTRUMENT PROBES OR LEADS TO THE POWER LINE OR TO AN OUTLET.
- DO NOT TRY TO FIX/IMPROVE AN INSTRUMENT BY YOURSELF.
- DO NOT POWER UP AN INSTRUMENT WHICH IS NOT WORKING OR DISASSEMBLED.
- DO NOT TOUCH A DISASSEMBLED OR PARTIALLY DISASSEMBLED INSTRUMENT EVEN IF IT IS NOT POWERED.
- WEAR PROTECTIVE GOGGLES EVERY TIME YOU USE A SOLDERING IRON.
- TO AVOID EXPLOSIONS, NEVER USE A SOLDERING IRON ON A POWERED CIRCUIT AND BATTERIES.
- PLACE A FAN TO DISPERSE SOLDER VAPORS FOR LONG PERIOD OF SOLDERING WORK.

5.9.1 Pre-Laboratory Problems

1. Find the current I in the circuit shown below and confirm the two Kirchhoff's Laws



2. Find the equivalent Thévenin circuit for the following circuit respect to the point A and B :



3. A voltage generator with voltage $V_0 = 10$V and internal resistance $R_0 = 10\Omega$ can deliver a maximum current $I_0 = 50$mA. Using a series of 4 resistors, we want to produce the voltage differences $V_1 = 9$V, $V_2 = 5$V and $V_3 = 3$V measured from the negative pole of the generator and using the maximum deliverable current. Calculate the resistances of the 4 resistors.

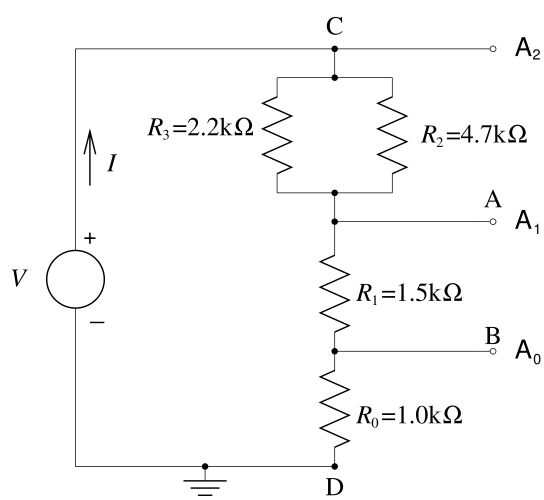
4. A DAQ system uses a 10bit Analog to Digital Converter (ADC) with a dynamic range from 0V to 5.115V. What is the resolution ΔV and the statistical uncertainty σ_V of the ADC? Assume that the converted voltage follows the uniform statistical distribution.

5.9.2 Procedure

Read completely the procedure before starting the circuit assembly, and taking measurements to avoid repeating parts of the procedure.

Ohm's and Kirchhoff's Laws

Set up the circuit shown below, using the nominal values of the resistors



Using a DMM or the DAQ system, do the following points:

1. Measure the resistances R_0, R_1, R_2, R_3 and R_{tot} seen the from the points **C** and **D**. Verify the resistance series and parallel laws for R_{tot}.

2. Verify Kirchhoff's voltage law for a loop containing the voltage source.

3. Verify Kirchhoff's current law for one of the nodes.

4. Using the DAQ system, and varying the voltage V, verify Ohm's law for the resistor R_1. Obtain the value of R_1 by fitting a properly taken set of data .

5. Connect a resistive load with resistance R_L "ad libitum" between **A** and **B**, and measure the voltage difference across it.

Thévenin Equivalent Circuit

Using the previous circuit do the following points:

1. Using a proper connection of the resistors of your circuit, build the Thévenin equivalent circuit seen from terminals **A** and **B**.
2. Verify the equivalence with the previous circuit by connecting the same resistive load R_L you used before.

Ideal Voltage Source

Using the available voltage source do the following points:

1. Determine the internal resistance of the source.
2. Find for which interval of the resistive load, the voltage source is ideal within 5% (i.e. the maximum voltage difference across the load 5% of voltage difference with no load).

5.10 Second Laboratory Week

Sections 4, and 6 of this chapter are required to complete the first laboratory week.

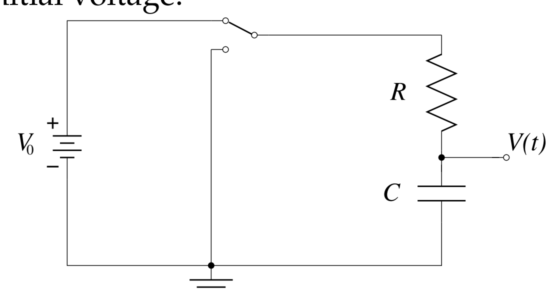
Depending on the application, semiconductor diodes can have quite different parameters. Silicon diodes used in the laboratory typically have $\eta \simeq 2$ for currents below ~ 100mA, breakdown voltage $V_b \simeq -50$V, and reverse saturation current $I_0 \sim 10\mu$A.

5.10.1 Pre-Laboratory Problems

1. Consider the RC circuit made of the series of a capacitor with capacitance C and a resistor of resistance R. Demonstrate that voltage $V(t)$ across capacitor, which is discharged through the resistor satisfies the following equation

$$V(t) = V_0 e^{-t/\tau}, \qquad \tau = RC,$$

where V_0 is the initial voltage.



2. Choose the values of R and C, with $R \ll 1\text{M}\Omega$ to get the time constant to be $\tau = 1$s. Prove that τ has the dimension of a time (hint: use eq.(5.2) and (5.1)).

3. With $\tau = 1\mu$s determine the time t^* for the voltage $V(t)$ to become less than 1% of the initial value V_0. Supposing that the sampling rate of a data acquisition system is 1000 samples/s, what is the minimum value of τ necessary to measure a variation of 1% of $V(t)$?

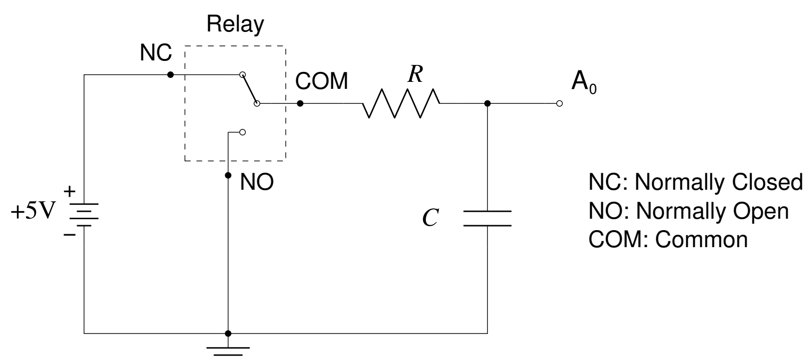
4. In semi-conductor materials like germanium and silicon the number of charge carriers n, strongly depends on the absolute temperature T, i.e. $n(T) = n_0 e^{-E_g/k_b T}$, where $k_b = 8.6 \cdot 10^{-5}$eV/K is the Boltzmann

constant and E_g is the gap from the conduction band and the valence band. Given $E_g = 0.67$eV for germanium, calculate the ratio of n at body temp to n at room temp. If $R \propto 1/n$, what is the ratio of the R at the two temperatures? Repeat for silicon where $E_g = 1.1$eV.

5.10.2 Procedure

RC Circuit Time Constant

1. Determine the time constant τ of the RC circuit using the following set-up



   Choose R and C such that $\tau \simeq 1$s. The uncertainty on the time is $\sigma_t = 2$ms.

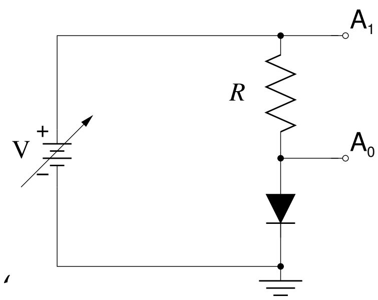
2. Compare the value of τ obtained in point one with the RC value obtained measuring directly the resistance and the capacitor.

3. Rearranging the previous circuit, redo the time constant τ measurement for a charging capacitor .

4. Compare the two values of τ indirectly measured.

Semiconductor Diodes

Make the following circuit, choosing the value of the resistor to limit the maximum current to ~ 5mA, and answer the next points:



1. Determine the polarity and the turn on voltage of a germanium diode and a silicon diode .

2. Determine the Boltzmann constant in eV (electron-volt) units from the V-I characteristic of a silicon diode. Assume the parameter $\eta_{si} = 2.0 \pm 0.1$ for the silicon diode.

3. Assuming the previous measured value of the Boltzmann constant determine the parameter η_{ge} of a germanium diode.

4. Calculate the value of η_{ge} for the germanium diode considering the standard value of $k \simeq 1.381 \cdot 10^{-23}$J/K, and then compare it with the previous measured value of η_{ge}.

Chapter 6

Alternating Current Network Theory

In this chapter we will study the properties of electronic networks propagating sinusoidal currents (alternate currents/AC). In this case, voltage or current sources produce sinusoidal waves whose frequency can be ideally changed from 0 to ∞. The AC analysis of such circuits is valid once the network is at the steady state, i.e. when the transient behavior (such as that produced by closing or opening switches) is extinguished.

In general, if we have a sinusoidal signal (voltage, or current) applied to a circuit having at least one input and one output, we will expect a change in amplitude and phase at the output. The determination of these quantities for quite simple circuits can be very complex. It is indeed important to develop a convenient representation of sinusoidal signals to simplify the analysis of circuits in the AC regime.

6.1 Symbolic Representation of a Sinusoidal Signals, Phasors

A sinusoidal quantity (a sinusoidal current or voltage for example) ,

$$A(t) = A_0 \sin(\omega t + \varphi),$$

is completely characterized by the amplitude A_0, the angular frequency ω, and the initial phase φ. The phase φ corresponds to a given time shift t^* of the sinusoid ($\omega t^* = \varphi \;\Rightarrow t^* = \varphi/\omega$).

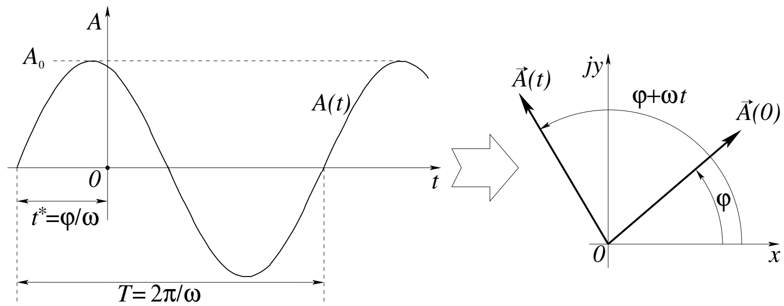


Figure 6.1: Sinusoidal quantity $A(t)$ and its phasor representation $\vec{A}$ at the initial time $t = 0$ and at time t.

We can indeed associate to $A(t)$ an applied vector $\vec{A}$ of the complex plane with modulus $|\vec{A}| = A_0 \geq 0$, rotating counter-clock wise around the origin O, with angular frequency ω, and initial angle φ (see figure 6.1). This vector is called *phasor.*

The complex representation of the phasor is indeed[1]

$$\vec{A} = A_0 e^{j(\omega t + \varphi)}, \qquad j = \sqrt{-1},$$

or

$$\vec{A} = x + jy, \qquad \begin{cases} x = A_0 \cos(\omega t + \varphi) \\ y = A_0 \sin(\omega t + \varphi) \end{cases}$$

Extracting the real and the imaginary part of the phasor, we can easily compute its amplitude A_0 and phase φ, i.e.

$$|\vec{A}| = \sqrt{\Re[\vec{A}]^2 + \Im[\vec{A}]^2} \qquad \varphi = \arg\left[\vec{A}\right] = \arctan\left(\frac{\Im[\vec{A}]}{\Re[\vec{A}]}\right), \quad (t = 0)$$

and reconstruct the real sinusoidal quantity. It is worthwhile to notice that in general, amplitude A_0 and phase φ are functions of the frequency.

The convenience of this representation will be evident, once we consider the operation of derivation and integration of a phasor.

[1]To avoid confusion with the symbol of the electric current i, it is convenient to use the symbol j for the imaginary unit.

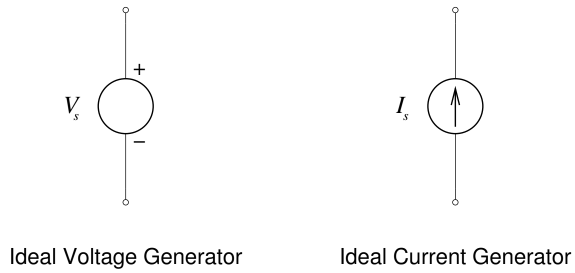


Figure 6.2: Ideal voltage and current generators symbols.

6.1.1 Derivative of a Phasor

Computing the derivative of a phasor $\vec{A}$, we get

$$\frac{d\vec{A}}{dt} = j\omega A_0 e^{j(\omega t+\varphi)} = j\omega \vec{A},$$

i.e. the derivative of a phasor is equal to the phasor times $j\omega$.

6.1.2 Integral of a Phasor

The integral of a phasor $\vec{A}$ is

$$\int_{t_0}^{t} \vec{A}dt' = \frac{1}{j\omega}\left[A_0 e^{j(\omega t+\varphi)} - A_0 e^{j(\omega t_0+\varphi)}\right] = \frac{1}{j\omega}\vec{A} + \text{const.},$$

i.e. the integral of a phasor is equal to the phasor divided by $j\omega$ plus a constant. For the AC regime we can assume the constant to be equal to zero without loss of generality.

Symbols for ideal sinusoidal voltage and current generators are shown in figure 6.2.

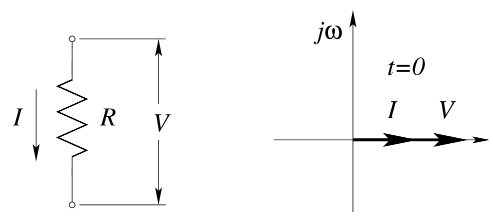
6.2 Current Voltage Equation for Passive Ideal Components with Phasors

Let's rewrite the I-V characteristic for the passive ideal components using the phasor notation. For sake of simplicity, we remove the arrow above the phasor symbol. To avoid ambiguity, we will use upper case letters to indicate phasors, and lower case letters to indicate a generic time dependent signal.

6.2.1 The Resistor

For time dependent signals, Ohm's law for a resistor with resistance R is

$$v(t) = Ri(t).$$



Introducing the phasor $I = I_0 e^{j\omega t}$ (see figure above), we get

$$v(t) = RI_0 e^{j\omega t},$$

and in the phasor notation

$$V = R\,I.$$

Note that in this case, the frequency and time dependence is implicitly contained in the phasor current I.

6.2.2 The Capacitor

The variation of the voltage difference dv across a capacitor with capacitance C, and due to the amount of charge dQ, is

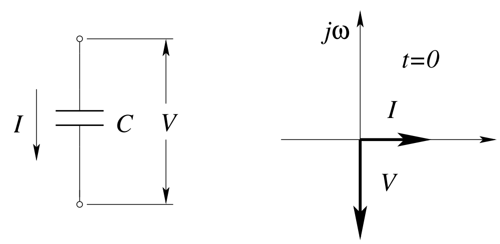
$$dv = \frac{dQ}{C}.$$

If the variation happens in a time dt, and considering that

$$i(t) = \frac{dQ}{dt},$$

we will have

$$\frac{dv(t)}{dt} = \frac{1}{C} i(t), \quad \Rightarrow \quad v(t) = \frac{1}{C}\int_0^t i(t')dt' + v(0).$$



Introducing the phasor $I = I_0 e^{j\omega t}$ (see figure above), we get

$$v(t) = \frac{1}{C}\int_0^t I_0 e^{j\omega t'} dt' + v(0),$$

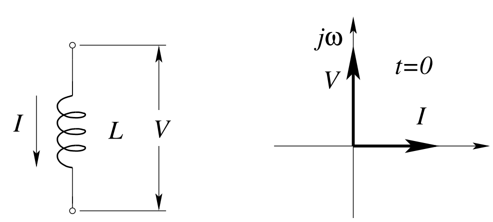
Using the phasor notation and supposing that for $t = 0$ the capacitor is discharged, we finally get

$$V = \frac{1}{j\omega C} I, \qquad , v(0) = 0.$$

6.2.3 The Inductor

The induced voltage $v(t)$ of an inductor with inductance L, is

$$v(t) = L\frac{di(t)}{dt}.$$



Introducing the phasor $I = I_0 e^{j\omega t}$ (see figure above), we get

$$v(t) = L\frac{d}{dt} I_0 e^{j\omega t},$$

and in the phasor notation

$$V = j\omega L\, I.$$

6.3 The Impedance and Admittance Concept.

Let's consider a generic circuit with a port, whose voltage difference and current are respectively the phasors $V = V_0 e^{j(\omega t+\varphi)}$, and $I = I_0 e^{j(\omega t+\psi)}$. The ratio Z between the voltage difference and the current

$$Z(\omega) = \frac{V}{I} = \frac{V_0}{I_0} e^{j(\varphi-\psi)}.$$

is said to be the *impedance of the circuit.*

The inverse

$$Y(\omega) = \frac{1}{Z(\omega)}$$

is called the *admittance of the circuit.*

For example, considering the results of the previous subsection, the impedance for a resistor, a capacitor, and an inductor are respectively

$$Z_R = R, \qquad Z_C(\omega) = \frac{1}{j\omega C}, \qquad Z_L(\omega) = j\omega L,$$

and the admittances are

$$Y_R = \frac{1}{R}, \qquad Y_C(\omega) = j\omega C, \qquad Y_L(\omega) = \frac{1}{j\omega L}.$$

In general, the impedance and the admittance of a circuit port is a complex function, which depends on the angular frequency ω. Quite often they are graphically represented by plotting their magnitude and phase .

6.3.1 Impedance in Parallel and Series

It can be easily demonstrated that the same laws for the total resistance of a series or a parallel of resistors hold for the impedance

$$Z_{tot} = Z_1 + Z_2 + ... + Z_N, \quad \text{(impedances in series)}$$

$$\frac{1}{Z_{tot}} = \frac{1}{Z_1} + \frac{1}{Z_1} + ... + \frac{1}{Z_N}, \quad \text{(impedances in parallel)}$$

6.3.2 Ohm's Law for Sinusoidal Regime

Thanks to the impedance concept, we can generalize Ohm's law and write the fundamental equation (*Ohm's law for sinusoidal regimes*)

$$V(\omega) = Z(\omega)I(\omega).$$

6.4 Two-port Network

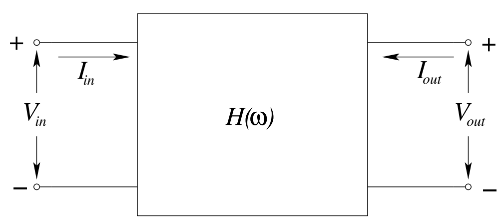


Figure 6.3: Two-port network circuit representation. The Voltage difference signs and current directions are conventional.

A linear circuit with one input and one output is called a two-port network (see figure 6.3).

To characterize the behavior of a two-port network, we can study the response of the output V_o as a function of the angular frequency ω of a sinusoidal input V_i.

In general, we can write

$$V_o(\omega) = H(\omega)V_i(\omega), \quad \text{or} \quad H(\omega) = \frac{V_o(\omega)}{V_i(\omega)},$$

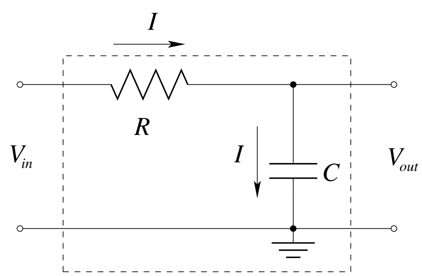


Figure 6.4: RC low-pass filter circuit.

where the complex function $H(\omega)$ is called the *transfer function of the two-port network*. The transfer function contains the information of how the amplitude and the phase of the input changes when it reaches the output. Knowing the transfer function of a two-port network, we characterize completely the circuit[2]. The definition of $H(\omega)$ suggests the way of measuring the transfer function. In fact, exciting the input with a sinusoidal wave we can measure the amplitude and the phase lead or lag respect to the input of the output signal.

To graphically represent $H(\omega)$, it is common practice to plot the magnitude $|H(\omega)|$ in a double logarithmic scale, and the phase $\arg[H(\omega)]$ in a semilogarithmic scale for the angular frequency.

It is important to notice that it is not necessary to have an ideal sinusoidal generator to make the transfer function measurement. In fact, if the input amplitude changes with the frequency the ratio between the output will not change. The same is true for the phase, i.e. if the input phase changes the difference with the output phase cannot change.

Let's study three common two port networks, the RC low-pass filter, the RC high-pass filter and the LCR series resonant circuit.

[2]A much deeper understanding of the transfer function , requires the concept of the Fourier transform and the Laplace transform.

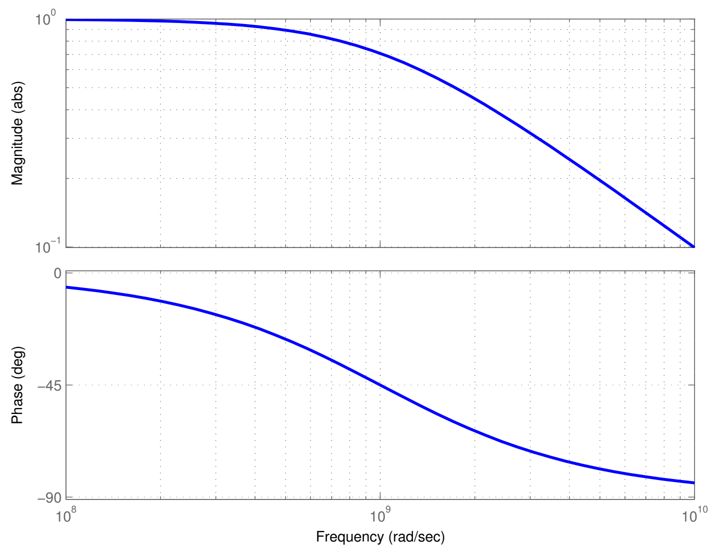


Figure 6.5: RC low-pass filter circuit transfer function.

6.4.1 The RC Low-Pass Filter

Figure 6.4 shows the *RC low-pass filter circuit*. The input and the output voltage differences are respectively[3]

$$\begin{aligned} V_{in} &= Z_{in} I = \left(R + \frac{1}{j\omega C} \right) I, \\ V_{out} &= Z_{out} I = \frac{1}{j\omega C} I, \end{aligned}$$

and the transfer function is indeed

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\tau\omega}, \qquad \tau = RC.$$

or

$$H(\omega) = \frac{1}{1 + j\omega/\omega_0}, \qquad \omega_0 = \frac{1}{RC}.$$

[3] V_{out} as function of V_{in} can be directly calculated using the voltage divider equation.

Computing the magnitude and phase of $H(\omega)$, we obtain

$$
\begin{aligned}
|H(\omega)| &= \frac{1}{\sqrt{1+\tau^2\omega^2}} \\
\arg(H(\omega)) &= -\arctan\left(\frac{\omega}{\omega_0}\right)
\end{aligned}
$$

Figure 6.5 shows the magnitude and phase of $H(\omega)$. The parameter τ and ω_0 are called respectively the *time constant* and the *angular cut-off frequency of the circuit*. The cut-off frequency is the frequency where the output V_{out} is attenuated by a factor $1/\sqrt{2}$.

It is worthwhile to analyze the qualitative behavior of the capacitor voltage difference V_{out} at very low frequency and at very high frequency.

For very low frequency the capacitor is an open circuit and V_{out} is essentially equal to V_{in}. for high frequency the capacitor acts like a short circuit and V_{out} goes to zero.

The capacitor produces also a delay as shown in the phase plot. At very low frequency the V_{out} follows V_{in} (they have the same phase). The output V_{out} loses phase ($\omega t = \varphi \Rightarrow t = \varphi/\omega$) when the frequency increases the frequency V_{out} starts lagging due to the negative phase ,φ, and then reaches a maximum delay due to a phase shift of $-\pi/2$.

6.4.2 The CR High-Pass Filter

Figure 6.6 shows the *CR high-pass filter circuit*. The input and the output voltage differences are respectively

$$
\begin{aligned}
V_{in} &= Z_{in}I = \left(R + \frac{1}{j\omega C}\right) I, \\
V_{out} &= Z_{out}I = RI,
\end{aligned}
$$

and indeed the transfer function is

$$
H(\omega) = \frac{V_{out}}{V_{in}} = \frac{j\omega\tau}{1+j\tau\omega}, \qquad \tau = RC.
$$

or

$$
H(\omega) = \frac{j\omega/\omega_0}{1+j\omega/\omega_0}, \qquad \omega_0 = \frac{1}{RC}.
$$

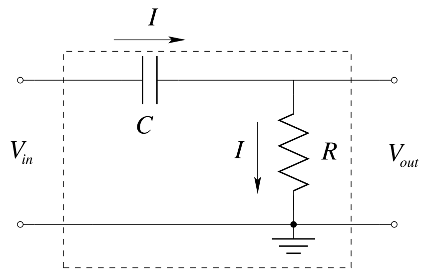


Figure 6.6: CR high pass filter circuit

Computing the magnitude and phase of $H(\omega)$, we obtain

$$
\begin{aligned}
|H(\omega)| &= \frac{\tau\omega}{\sqrt{1+\tau^2\omega^2}}, \\
\arg(H(\omega)) &= \arctan\left(\frac{\omega_0}{\omega}\right)
\end{aligned}
$$

Figure 6.7 shows the magnitude and phase of $H(\omega)$. The definitions in the previous subsection for τ, and ω_0 hold for the RC high-pass filter.

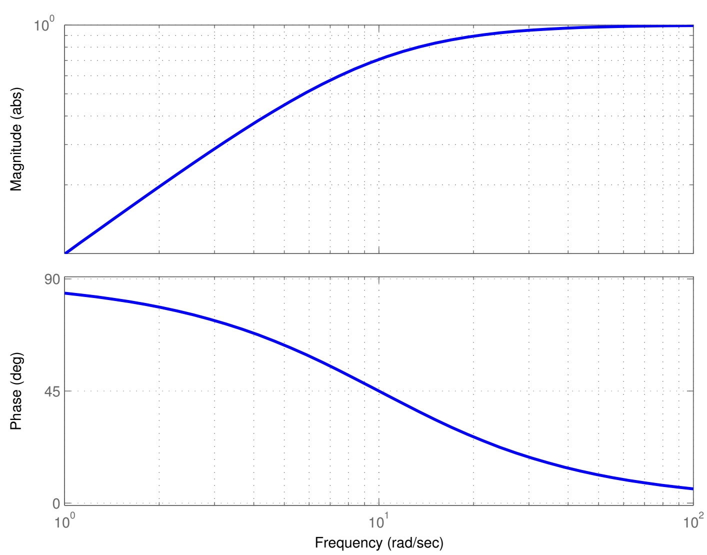


Figure 6.7: CR high pass filter circuit transfer function.

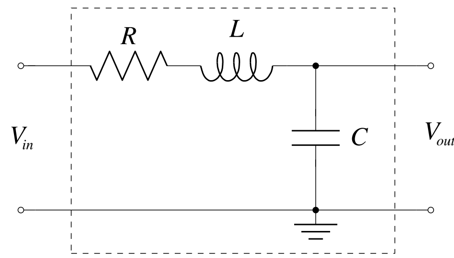


Figure 6.8: LCR series resonant circuit.

6.4.3 The LCR Series Resonant Circuit

Figure 6.8 shows the *LCR series circuit*. Considering the voltage difference across the capacitor as the circuit output, we will have

$$\begin{aligned} V_{in} &= \left(R + j\omega L + \frac{1}{j\omega C}\right) I, \\ V_{out} &= \frac{1}{j\omega C} I , \end{aligned}$$

and the transfer function will be

$$H_C(\omega) = \frac{1}{j\omega RC - \omega^2 LC + 1}.$$

Computing the magnitude and phase of $H(\omega)$, we obtain

$$\begin{aligned} |H_C(\omega)| &= \frac{1}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \\ \arg\left[H_C(\omega)\right] &= \arctan\left(\frac{\omega RC}{\omega^2 LC - 1}\right) \end{aligned}$$

For sake of simplicity It is convenient to define the two following quantities

$$\omega_0^2 = \frac{1}{LC}, \qquad Q = \frac{1}{R}\sqrt{\frac{L}{C}} .$$

The parameter Q is called the quality factor of the circuit. Considering the previous definitions, and after some algebra $H_C(\omega)$ can be rewritten as

$$H_C(\omega) = \frac{\omega_0^2}{-\omega^2 + j\omega\frac{\omega_0}{Q} + \omega_0^2} \,. \tag{6.1}$$

The magnitude has an absolute maximum for

$$\omega_C^2 = \omega_0^2 \left(1 - \frac{1}{2Q^2}\right), \qquad \text{(angular resonant frequency)}$$

and the maximum is

$$|H_C(\omega_C)| = \frac{Q}{\sqrt{1 - \frac{1}{2Q^2}}}, \qquad \text{if } Q \gg 1 \quad \Rightarrow \omega_C \simeq \omega_0,\ |H_C(\omega_C)| \simeq Q$$

Far from resonance the approximate behavior of $|H_C(\omega)|$ is

$$\begin{array}{lcl} \omega \ll \omega_C & \Rightarrow & |H_C(\omega)| \simeq 1 \\ \omega \gg \omega_C & \Rightarrow & |H_C(\omega)| \simeq \dfrac{\omega_0^2}{\omega^2} \end{array}$$

Figure 6.9 shows the magnitude and phase of $H_C(\omega)$.

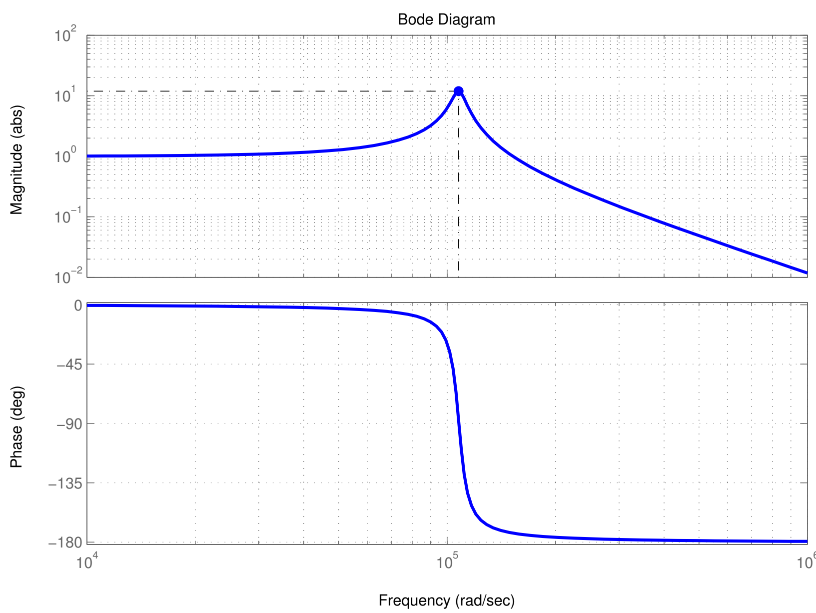


Figure 6.9: Transfer Function $H_C(\omega)$of the LCR series resonant circuit .

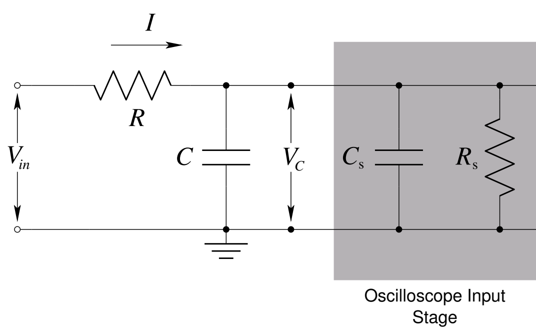
6.5 First Laboratory Week

6.5.1 Pre-laboratory Exercises

1. Calculate the total impedance of a series of a resistor with a capacitor and for a parallel of a resistor with a capacitor. Do your results confirm the statement that capacitors behave as a short circuit at high frequencies and as an open circuit at low frequencies ?

2. Calculate the magnitude of the total impedance for a series of a resistor with a capacitor having $R = 10\text{k}\Omega$, $C = 2.5\text{nF}$, and $\upsilon = 20\text{kHz}$. Calculate the same quantity for a parallel of a resistor with a capacitor having $R = 10\text{M}\Omega$, $C = 30\text{pF}$, and $\upsilon = 20\text{kHz}$.

3. The circuit shown below includes the impedance of the input channel of the CRT oscilloscope, and V_s is indeed the real voltage measured by the instrument.



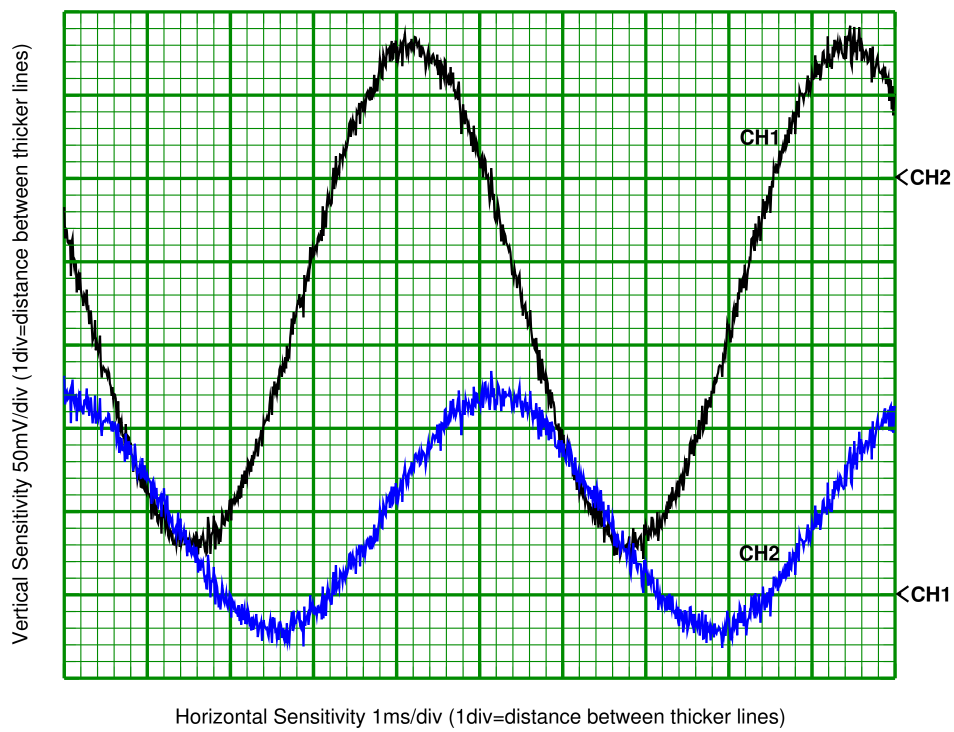
   Find the voltage $V_s(\omega)$, and the angular cut-off frequency ω_0 of the transfer function V_s/V_{in}(i.e. the value of ω for which $|V_s/V_{in}| = 1/\sqrt{2}$).
   Show that for $\omega = 0$ the $V_s(\omega)$ formula simplifies and becomes the resistive voltage divider equation.
   Demonstrate that the conditions to neglect the input impedance of the oscilloscope are the following :

$$C \gg C_s, \qquad R \ll R_s$$

4. Considering the previous circuit, calculate the value of R to obtain $V_{in} \simeq V_s$ with a fractional systematic error of 1%, if $\omega = 0\text{rad/s}$ and $R_s = 1\text{M}\Omega$.

5. Read the first two sections of the oscilloscope notes (see appendix B).

6. Considering the figure below (a "snapshot" of an oscilloscope display), determine the peak to peak amplitude, the DC offset, the frequency of the two sinusoidal curves, and the phase shift between the two curves (channels horizontal axis position is indicated by an arrow and the channel name on the right of the figure).



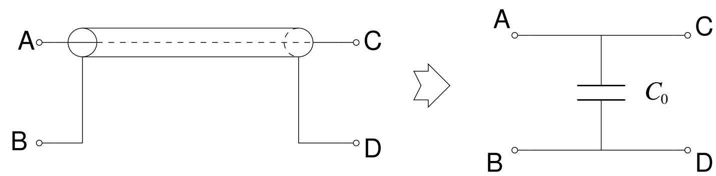
6.5.2 Procedure

Read carefully the text before starting the laboratory measurements.

Note that instead of the angular frequency ω, this procedure reports the frequency ν ($\omega = 2\pi\nu$), which is more often used in laboratory measurements.

"BNC" cables and wires terminated with "banana" connectors are available to connect circuits to the available instruments.

BNC[4] cables, a diffused type of radio frequency (RF) coaxial cable, have an intrinsic capacitance due to their geometry as shown in the figure below



They have typical linear density capacitance $\Delta C/\Delta l \sim 100\text{pF/m}$. Single wires have usually smaller capacitance than BNC cables. Unfortunately, their capacitance strongly depends on how they are positioned one respect to the others.

- Build a RC low-pass filter or a CR high-pass filter with a cut-off frequency ν_0 between 1kHz and 100kHz, using values for R and C, which makes the input impedance of the oscilloscope $Z_s$negligible compared to the impedance of your circuit.

- Measure the transfer function $H(\nu)$ of your circuit by measuring $|H(\nu)|$, and $\arg[H(\nu)]$. Then, fit the experimental data with the proper theoretical curves.

- Experimentally find $|H(\nu_0)|$ and $\arg[H(\nu_0)]$ (at the cut-off frequency ν_0), and compare them with the theoretical data.

- Build a RC low-pass filter using a capacitance C comparable with the input capacitance C_s of the oscilloscope, and check the perturbation induced by the instrument at the cut-off frequency ν_0.

6.6 Second Laboratory Week

6.6.1 Pre-laboratory Exercises

1. Considering an inductor made of a coil with large inductance ($L \sim 10\text{mH}$), resistance $R_L = 80\Omega$, wire diameter $d = 100\mu\text{m}$, and resis-

[4]"BNC" seems to stand for Bayonet Neill Concelman (named after Amphenol engineer Carl Concelman). Other sources claim that the acronym means British Navy Connector. What is certain is that the BNC connector was developed in the late 1940's as a miniature version of the type C connector (what does the "C" stand for ?)

tivity of $\rho \simeq 16\text{n}\Omega \cdot \text{m}$ (copper), determine the length l of the coil wire.

2. Demonstrate that the magnitude of the LCR series transfer function $H_C(\omega)$ has a maximum for $\omega = \omega_C$ and the maximum is equal to the quality factor Q, ($Q \gg 1$).

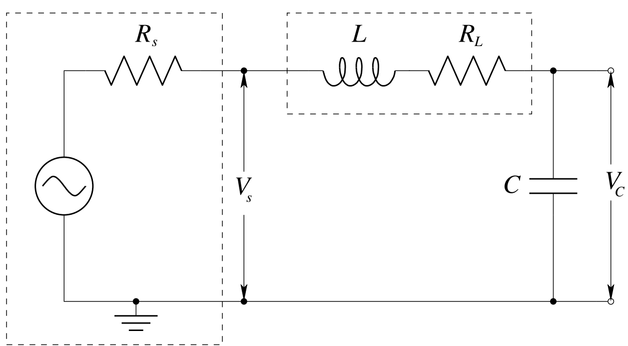
3. Supposing that R, ω_0, and Q of an LCR series circuit are known determine L, and C.

4. Compute the Magnitude and phase of equation (6.1).

5. Determine the phase of the LCR series transfer function $H_C(\omega)$ at $\omega = 0$, at the angular resonant frequency ω_C, and for $\omega \to \infty$.

6. Determine the capacitance C of a LCR series circuit necessary to have a resonant frequency $\nu_C = 20$kHz if $L = 10$mH, and $R = 80\Omega$. Then, calculate the quality factor of the LCR series circuit.

6.6.2 Procedure



- Construct a LCR series resonant circuit above with a resonant frequency ν_C of about 20kHz. Note that the resistance R_s is the internal resistance of the function generator, and the resistance R_L is the resistance of the inductor. Use a mylar capacitor to make the circuit.

- Measure magnitude and phase of the transfer function

$$H_C(\nu) = \frac{V_C}{V_s}.$$

- Fitting the magnitude and phase of H_C, determine the frequency ν_0 and the quality factor Q .
- Experimentally find the resonant frequency ν_C where the maximum of $|H_C(\nu)|$ occurs and compare it with the value indirectly obtained from the fits.
- Experimentally find the resonant frequency ν_C where the proper phase shift for the resonant frequency occurs and compare it with the value obtained from the fits.
- Directly measuring the values of R, L, and C, calculate ν_0, and Q and compare with the values obtained from the fits.
- Assuming that the direct measurement of R is correct, using ν_0 and Q values obtained from the fits, estimate L and C.

Appendix A

Data Acquisition System "Experimenter"

The "Experimenter" device is a very simple data acquisition system (DAQ) which allows to acquire several channels with a low data rate of 20samples/s. This are its main characteristics:

- maximum sampling frequency $\nu_s = 100$samples/s
- Timer resolution $\Delta T = 2ms$
- **Input channels Characteristics**
  - Number : 4
  - Number of bits: $n = 10$
  - range : unipolar from 0V to 5.115V $\quad \Rightarrow \Delta V_i = 5.115V$
  - resolution $\delta V_i = \Delta V_i / 2^n = 5$mV
- **Output Channels Characteristics**
  - Number: 2
  - Number of bits: $n = 8$
  - type : unipolar
  - range from 0 to 5V, $\quad \Rightarrow \Delta V_o = 5V$
  - resolution: $\delta V_o = \Delta V_o / 2^n \simeq 20$mV (not linear)
  - maximum current output: 1mA

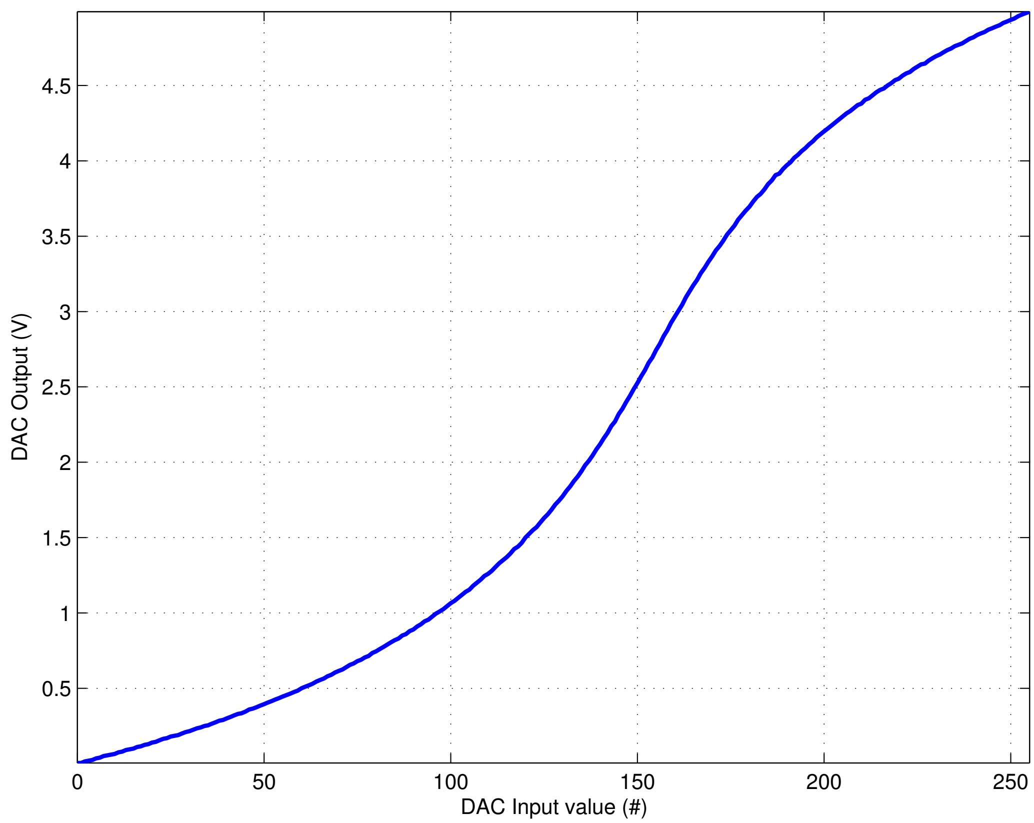


Figure A.1: Output channel characteristics

A.1 Output Channel Characteristics

The output channel control is implemented using the so called pulse with modulation (PWM) technique . Essentially, a PWM voltage source works by changing the time that the pulse is on (the duty cycle), and sending the pulse to a low pass filter.

The non linearity of the output channel is clearly shown in the plot shown in figure A.1. The reason of such non-linearity comes from the way that PWM is implemented.

A.2 ExperTerm Program

The “ExperTerm” is a terminal interface to communicate via RS232 port to the DAQ. This is the list of the available commands:

- **a ch0 [ch1]** ... print ADC sampled value of channel ch0 ch1 ... Example: a 1
- **A ch0 [ch1]** ... continuously print ADC sampled values of channel ch0 ch1 ... until a ctrl c is pressed. Example: A 1 0 3
- **e ch val** set voltage source "ch" to "val". Example: e 0 124
- **h** print this help screen q quit program

A.3 ExperDAQ Program

The “ExperDAQ” is a command line program for continuous data acquisition with the Experimenter. It allows to specify the acquisition parameters (input channels, number of samples, sampling frequency) to change internal relay status, and to set one voltage source. The following text is the program usage help:

usage: ExperDAQ **[-v]** **[-p Port,BaudRate,Parity,Bits,StopBit]** **[-o Channel,A,B,N]** **[-r]** **ChannelList Samples SamplingRate Averages FileName**

Parameters between square brackets are optional.

- **-v** verbose mode.
- **-p** serial port configuration example: -p 1,9600,0,8,1
- **-o Channel,A,B,N** source output settings. Trailing spaces are not allowed
  - **Channel**: 0 or 1
  - **A:** initial value from 0 to 255
  - **B:** final value from 0 to 255

  - **N:** number of step to go from A to B. The step is done every 1/SampleFrequency

- **-r** change relay status before data acquisition and restore previous status after the acquisition
- **ChannelList:** list of channels to acquire from 0 to 7. Values are separated with a comma. Trailing spaces are not allowed. Example: 3,0,1
- **Samples number:** samples to acquire
- **SamplingRate:** samples per seconds to acquire from 1 to 20
- **Averages:** number of averages. 0 for no averages. The data used to compute the average are collected at the SamplingRate
- **FileName:** filename containing the acquired data. First column is the time. The other columns are the channels values specified in the ChannelList

Example:

ExperDAQ -r -o0,0,255,20 0,1,2 20 10 0 data.txt

The previous example does the following: change relay status during the acquisition, sweep the voltage source 0 from 0 to 255, acquires 20 samples of ADC channels 0,1,2 with a sampling rate of 10sample/s, does not perform averages, and finally saves the data into the file data.txt.

Appendix A

The Vernier

The vernier[1] is essentially a pair of linear scales (see figure A.2), one the auxiliary scale which slides parallel along the other scale, the main scale, and acts like a "mechanical scale magnifier". Essentially, it allows to read fractions $1/n$ of the main scale divisions using the maximum possible *resolution* of the scale.

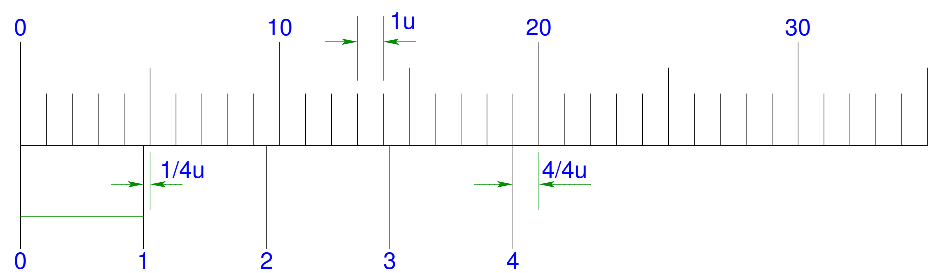


Figure A.1: Example of a vernier with $k = 5, n = 4$,and $\delta x = 1/4$units. The two scales are aligned and the 4-th auxiliary division is aligned to the $kn - 1 = 19$-th division of the main scale.

Each auxiliary scale division is $k - 1/n$ shorter than k divisions of the main scale. This means that if we align the first auxiliary scale division to the k-th division of the main scale, the distance x between the origin of the two scales will be $x = 1/n$units. If we align the second division

[1]Pierre Vernier XVII sec. mathematician and scientist inventor of the so called vernier caliper.

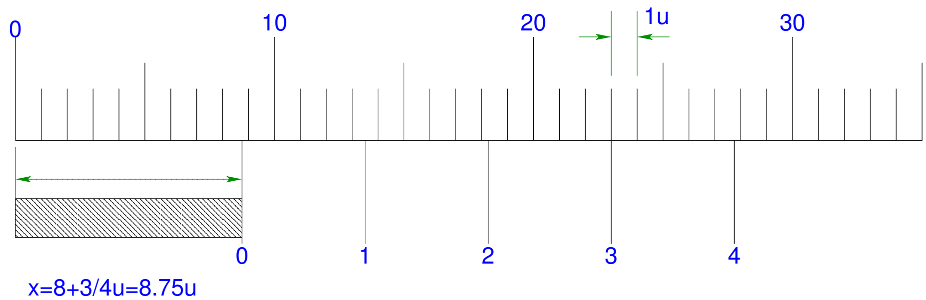


Figure A.2: Example of a vernier measurement with $k = 5, n = 4 \Rightarrow \Delta x = 1/4$units.

then $x = 2/n$units. In general, aligning the m-th division of the auxiliary scale to the m times k-th division of the main scale , will make the distance between the origin of the two scales equal to $x = m/n$.

A.1 Measuring with a Vernier

If we perform a measurement of a physical quantity x (see figure A.2) and we have the auxiliary scale zero after the N division of the main scale and the m-th auxiliary scale division is the division aligned to one of the main scale divisions, we will have

$$x = \left(N + \frac{m}{n}\right) \text{units} .$$

Because we cannot identify an interval were the measurement lies in, the read error δx to assign to x is

$$\delta x = \frac{1}{n}\text{units} .$$

In the example of figure A.2, we have

$$\begin{cases} N &= 8 \\ m &= 3 \\ n &= 4 \end{cases} \qquad \Rightarrow \qquad x = \left(8 + \frac{3}{4}\right) = 8.75 = (8.8 \pm 0.3)\,\text{units} .$$

The $1/n$ factor and the units should be printed out on the vernier.

The resolution limit of a vernier, depends on the *accuracy*[2] of tracing both scales. If we have more than one trace coincident, the instrument claimed resolution is illusory.

Calipers are one of the most common instruments that uses a vernier.

A.2 Probability Density Function Using a Vernier

Let's suppose that we use an instrument with a vernier to measure a physical quantity x. In this case, we cannot assume that x follows a uniform PDF as we did for measurements performed using single linear scales. In fact, because we look for the best aligned divisions pair to perform the measurement, we cannot identify an interval were the measurements lies in. If we could, it means that the vernier is not properly designed or manufactured. In other words, the instrument is practically unusable at least with the resolution claimed by the manufacturer.

[2]The resolution depends on the technology used to manufacture the two scales. Moreover, the dynamic range limits the instrument resolution because it is hard to keep a given accuracy for an arbitrary length of the scale.

Appendix B

The Cathode Ray Tube Oscilloscope

B.1 The Cathode Ray Tube Oscilloscope

The *cathode ray tube oscilloscope* is essentially an analog[1] instrument that is able to measure time varying electric signals. It is made of the following functional parts (see figure B.1):

- the cathode ray tube (CRT),
- the trigger,
- the horizontal input,
- the vertical input,
- time base generator.

Let's study in more detail each component of the oscilloscope.

B.1.1 The Cathode Ray Tube

The CRT is a vacuum envelope hosting a device called *an electron gun* , capable of producing an electron beam, whose transverse position can be modulated by two electric signals (see figures B.1 and B.7).

[1]Hybrid instruments combining the characteristics of digital and analog oscilloscopes, with a CRT, are also commercially available.

When the electron gun cathode is heated by wire resistance because of the Joule effect it emits electrons . The increasing voltage differences between a set of shaped anodes and the cathode accelerates electrons to a terminal velocity v_0 creating the so called electron beam.

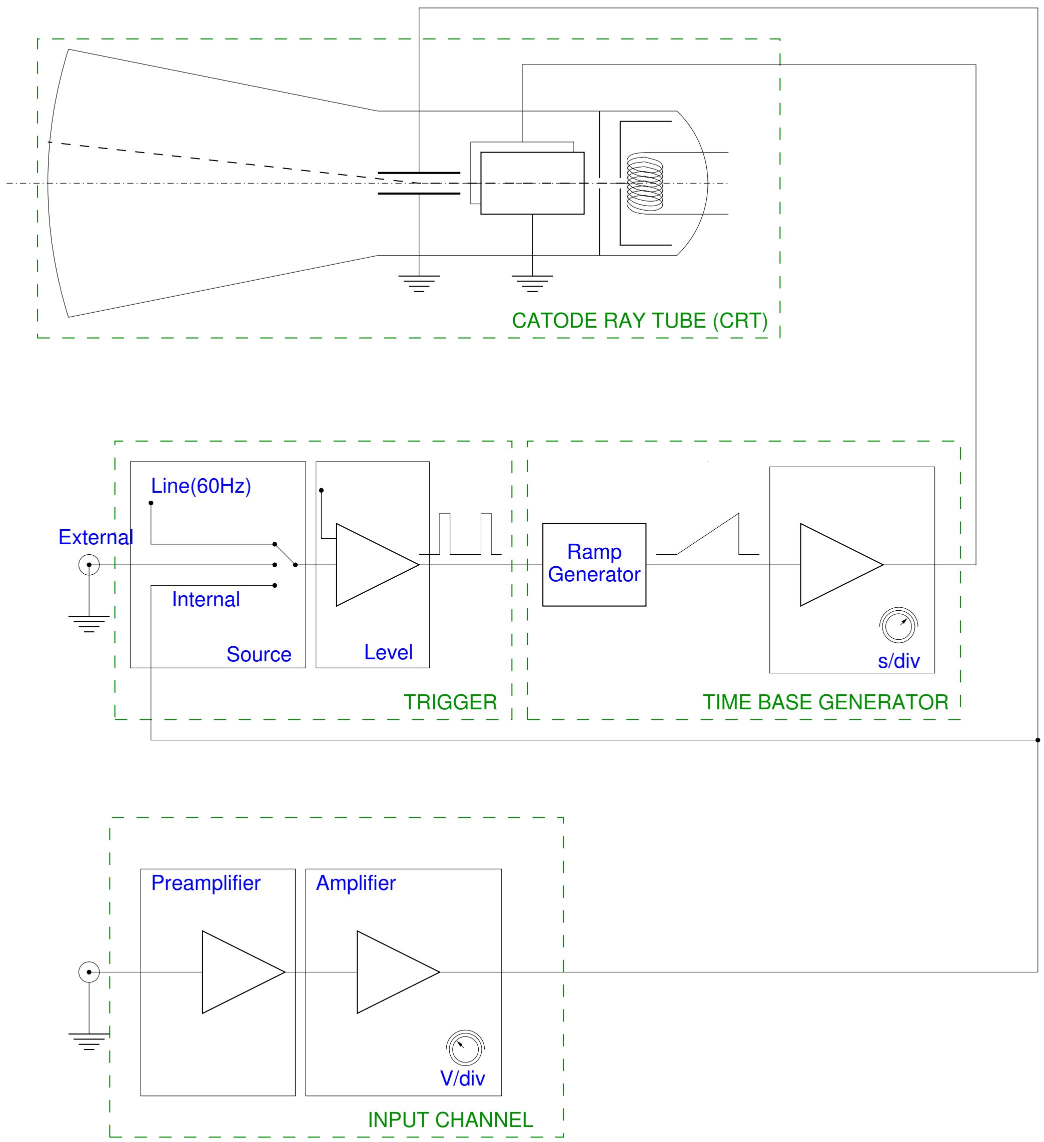


Figure B.1: Oscilloscope functional schematics

The beam then goes through two orthogonally mounted pairs of metallic plates. Applying voltage difference to those plates V_x and V_y, the beam is deflected along two orthogonal directions (x and y) perpendicular to its direction z. The deflected electrons will hit a plane screen perpendicular to the beam and coated with florescent layer. The electrons interaction with this layer generates photons, making the beam position visible on the screen.

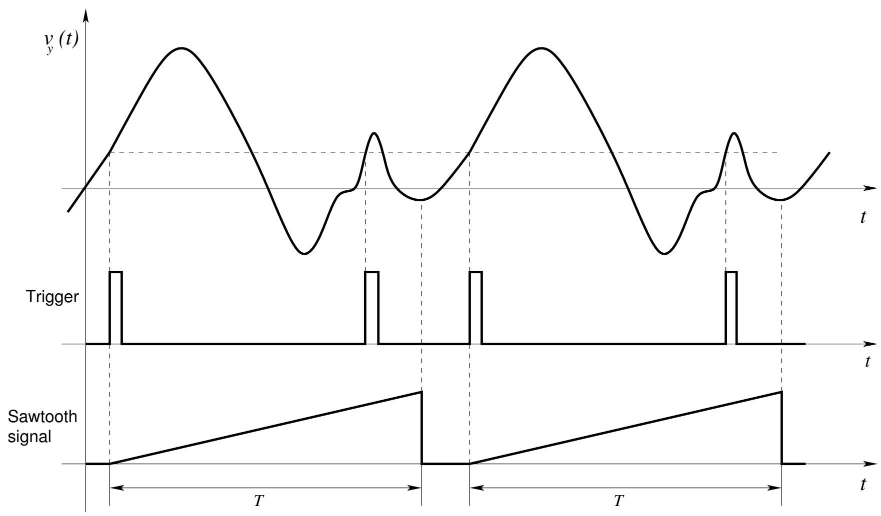


Figure B.2: Periodic Signal triggering.

B.1.2 The Horizontal and Vertical Inputs

The vertical and horizontal plates are independently driven by a variable gain amplifier to adapt the signals $v_x(t)$, and $v_y(t)$ to the screen range. A DC offset can be added to each input to position the signals on the screen. These two channels used to drive the signals to the plates signals are called horizontal and vertical inputs of the oscilloscope.

In this configuration the oscilloscope is an x-y plotter.

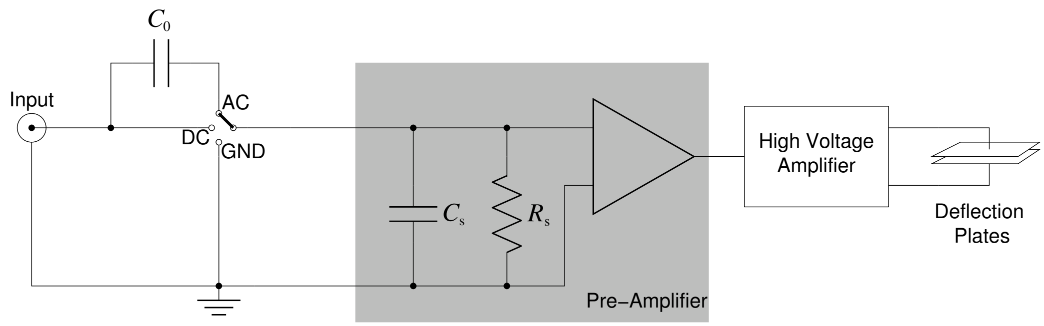


Figure B.3: Oscilloscope input impedance representation using ideal components (gray box). Input channel coupling is also shown.

B.1.3 The Time base Generator

If we apply a sawtooth signal $V_x(t) = \alpha t$ to the horizontal input, the horizontal screen axis will be proportional to time t. In this case a signal $v_y(t)$ applied to the vertical input, will depict on the oscilloscope screen the signal time evolution.

The internal ramp signal is generated by the instrument with an amplification stage that allows changes in the gain factor α and the interval of time shown on the screen. This amplification stage and the ramp generator are called the *time base generator.*

In this configuration, the horizontal input is used as a second independent vertical input, allowing the plot of the time evolution of two signals.

Visualization of signal time evolution is the most common use of an oscilloscope.

B.1.4 The Trigger

To study a periodic signal $v(t)$ with the oscilloscope, it is necessary to synchronize the horizontal ramp $V_x = \alpha t$ with the signal to obtain a steady plot of the periodic signal. The trigger is the electronic circuit which provides this function. Let's qualitatively explain its behavior.

The trigger circuit compares $v(t)$ with a constant value and produces a pulse every time the two values are equal and the signal has a given slope. The first pulse triggers the start of the sawtooth signal of period[2] T , which will linearly increase until it reaches the value $V = \alpha T$, and then is reset to zero. During this time, the pulses are ignored and the signal $v(t)$ is plotted for a duration time T. After this time, the next pulse that triggers the sawtooth signal will happen for the same previous value and slope sign of $v(t)$, and the same portion of the signal will be re-plotted on the screen.

B.2 Oscilloscope Input Impedance

A good approximation of the input impedance of the oscilloscope is shown in the circuit of figure B.3. The different input coupling modes (DC AC GND) are also represented in the circuit.

The amplifying stage is modeled using an ideal amplifier (infinite input impedance) with a resistor and a capacitor in parallel to the amplifier input.

The switch allows to ground the amplifier input and indeed to vertically set the origin of the input signal (GND position), to directly couple the input signal (DC position), or to mainly remove the DC component of the input signal (AC position).

B.3 Oscilloscope Probe

An oscilloscope probe is a device specifically designed to minimize the capacitive and resistive load added when the instrument is connected to the circuit. The price to pay is an attenuation of the signal that reaches the oscilloscope input[3].

Let's analyze the behavior of a passive probe. Figure B.4 shows the schematics of the equivalent circuit of a passive probe and of the input stage of an oscilloscope. The capacitance of the probe cable can be considered included in C_s

Considering the voltage divider equation, we have

$$H(j\omega) = \frac{V_s}{V_i} = \frac{Z_s}{Z_p + Z_s}, \tag{B.1}$$

[2]In general, the sawtooth signal period T and the period of $v(t)$ are not equal.
[3]Active probes can partially avoid this problems by amplifying the signal.

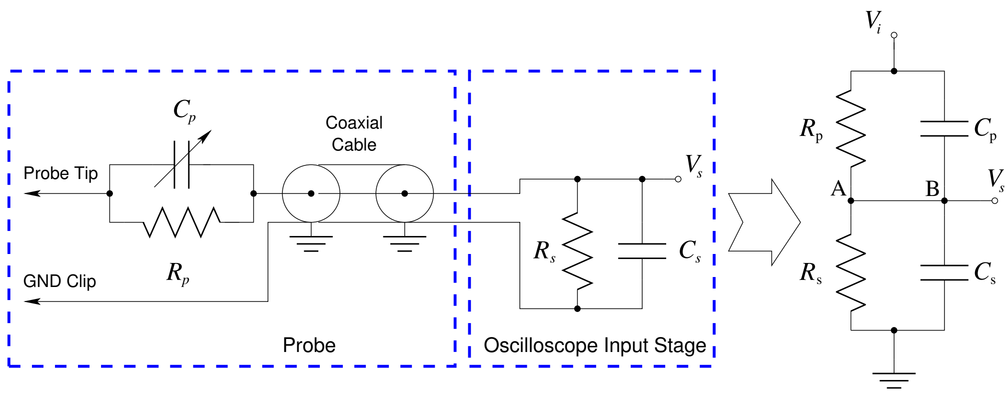


Figure B.4: Oscilloscope input stage and passive probe schematics.The equivalent circuit made of ideal components for the probe shielded cable is not shown.

where

$$\frac{1}{Z_s} = j\omega C_s + \frac{1}{R_s}, \qquad \frac{1}{Z_p} = j\omega C_p + \frac{1}{R_p},$$

and then

$$Z_s = \frac{R_s}{j\omega\tau_s + 1}, \qquad Z_p = \frac{R_p}{j\omega\tau_p + 1}.$$

Defining the following parameters

$$\tau_p = C_p R_p, \quad \alpha = \frac{R_s}{R_s + R_p}, \qquad \beta = \frac{C_p}{C_s + C_p},$$

and after some tedious algebra, equation (B.1) becomes

$$H(j\omega) = \alpha \frac{1 + j\omega\tau_p}{1 + j\omega\frac{\alpha}{\beta}\tau_p},$$

which is the transfer function from the probe input to the oscilloscope input before the ideal amplification stage.

The DC and high frequency gain of the transfer function $H(j\omega)$ are respectively

$$H(0) = \alpha, \qquad H(\infty) = \beta.$$

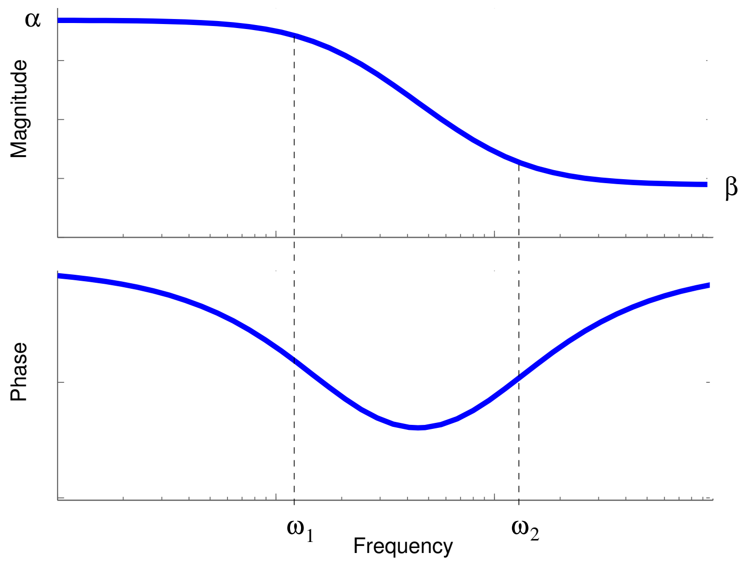


Figure B.5: Qualitative transfer function from the under compensated probe input to the oscilloscope input before the ideal amplification stage. As usual, the oscilloscope input is described having an impedance $R_s||C_s$.

The numerator and denominator of $H(j\omega)$ are respectively equal to zero, (the zeros and poles of H) when

$$\omega = \omega_z = j\frac{1}{\tau_p}, \qquad \omega = \omega_p = j\frac{\beta}{\alpha}\frac{1}{\tau_p}.$$

Figure B.5 shows the qualitative behavior of H for $\frac{\alpha}{\beta} > 1$.

B.3.1 Probe Frequency Compensation

By tuning the variable capacitor C_p of the probe, we can have three possible cases

$$\frac{\alpha}{\beta} < 1 \quad \Rightarrow \quad \text{over-compensation}$$
$$\frac{\alpha}{\beta} = 1 \quad \Rightarrow \quad \text{compensation}$$

$$\frac{\alpha}{\beta} > 1 \quad \Rightarrow \quad \text{under-compensation}$$

if $\alpha < \beta$ the transfer function attenuates more at frequencies above ω_z, and the input signal V_i is distorted.

if $\alpha = \beta$ the transfer function is constant and the input signal $V_i$will be undistorted, and attenuated by a factor α.

if $\alpha > \beta$ the transfer function attenuates more at frequencies below ω_p and the input signal $V_i$is distorted.

The ideal case is indeed the compensated case, because we will have increased the input impedance by a factor α without distorting the signal.

The probe compensation can be tuned using a signal, which shows a clear distortion when it is filtered. A square wave signal is very useful in this case because, it shows a different distortion if the probe is under or over compensated. Figure B.6 sketches the expected square wave distortion for the two un-compensated cases.

It is worthwhile to notices that

$$\frac{\alpha}{\beta} = 1, \qquad \Rightarrow \frac{R_s}{R_p} = \frac{C_p}{C_s}.$$

This condition implies that:

- the voltage difference V_1 across R_s is equal the voltage difference V_2 across C_s, i.e $V_1 = V_2$
- the voltage difference V_3 across R_p is equal the voltage difference V_4 across C_p , i.e. $V_3 = V_4$
- and indeed $V_1 + V_2 = V_3 + V_4$.

This means that no current is flowing through the branch AB, and we can consider just the resistive branch of the circuit to calculate V_s. Applying the voltage divider equation, we finally get

$$V_s = \frac{R_s}{R_s + R} V_i$$

The capacitance of the oscilloscope does not affect the oscilloscope input anymore, and the oscilloscope+probe input impedance R_i becomes greater, i.e.

$$R_i = R_s + R_p.$$

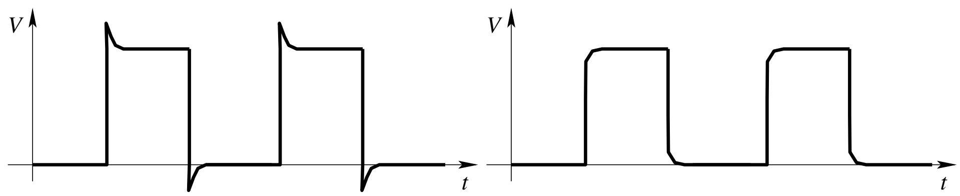


Figure B.6: Compensation of a passive probe using a square wave. Left figure shows an over compensated probe, where the low frequency content of the signal is attenuated. Right figure shows the under compensated case, where the high frequency content is attenuated.

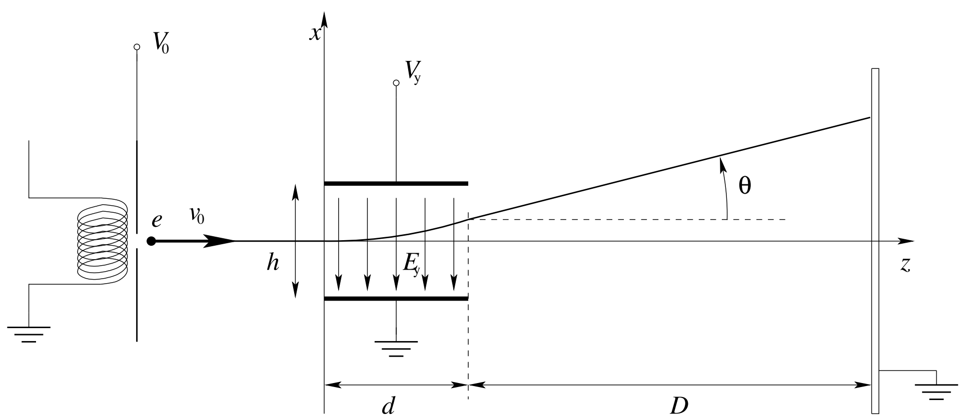


Figure B.7: CRT tube schematics. The electron enters into the electric field and makes a parabolic trajectory. After passing the electric field region it will have a vertical offset and deflection angle θ.

B.4 Beam Trajectory

Let's consider the electron motion through one pair of plates.

The electron terminal velocity v_0 coming out from the gun can be easily calculated considering that its initial potential energy is entirely converted into kinetic energy, i.e

$$\frac{1}{2}\mu v_0^2 = eV_0, \quad \Rightarrow \quad v_0 = \sqrt{2\frac{eV_0}{\mu}},$$

where μ is the electron mass, e the electron charge, and V_0 the voltage applied to the last anode.

If we apply a voltage V_y to the plates whose distance is h, the electrons will feel a force $F_y = eE_y$ due to an electric field

$$|E_y| = \frac{V_y}{h}.$$

The equation of dynamics of the electron inside the plates is

$$\begin{aligned} \mu\ddot{z} &= 0, \quad \Rightarrow \quad \dot{z} = v_0, \\ \mu\ddot{y} &= e|E_y|. \end{aligned}$$

Supposing that V_y is constant, the solution of the equation of motion will be

$$\begin{aligned} z(t) &= \sqrt{2\frac{eV_0}{\mu}}t, \\ y(t) &= \frac{1}{2}\frac{eV_y}{\mu h}t^2. \end{aligned}$$

Removing the dependency on the time t, we will obtain the electron beam trajectory , i.e.

$$y = \frac{1}{4h}\frac{V_y}{V_0}z^2,$$

which is a parabolic trajectory.

Considering that the electron is transversely accelerated until $z = d$, the total angular deflection θ will be

$$\tan\theta = \left(\frac{\partial y}{\partial z}\right)_{z=d} = \frac{1}{2}\frac{d}{h}\frac{V_y}{V_0}.$$

and displacement Y on the screen is

$$Y(V_y) = y(z=d) + \tan\theta D,$$

i.e.,

$$Y(V_y) = \frac{1}{2}\frac{d}{h}\frac{1}{V_0}(\frac{d}{2} + D)V_y.$$

Y is indeed proportional to the voltage applied to the plates through a rather complicated proportional factor.

The geometrical and electrical parameters of this proportional factor play a fundamental role in the resolution of the instrument. In fact, the smaller the distance h between the plates, or the smaller the gun voltage drop V_0, the larger is the displacement Y. Moreover, Y increases quadratically with the electron beam distance d.

B.4.1 CRT Frequency Limit

The electron transit time through the plates determine the maximum frequency that a CRT can plot. In fact, if the transit time τ is much smaller than the period T of the wave form $V(t)$, we have

$$V(t) \simeq \text{constant}, \qquad \text{if} \quad \tau \ll T,$$

and the signal is not distorted.

The transit time is

$$\tau = \frac{d}{v_0} = d\sqrt{\frac{\mu}{2eV_0}}.$$

Supposing that

$$\begin{cases} V_0 = 1\text{kV} \\ d = 20\text{mm} \\ \mu c^2 \simeq 0.5\text{MeV} \\ e = 1\text{eV} \end{cases} \qquad \Rightarrow \qquad \tau \simeq 1\text{ns}$$